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# A sharp adaptive confidence ball for self-similar functions

Richard Nickl[a,*], Botond Szabó[b,c]

[a] University of Cambridge, United Kingdom
[b] University of Amsterdam, Netherlands
[c] Budapest University of Technology and Economics, Hungary

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**Abstract**

In the nonparametric Gaussian sequence space model an $\ell^2$-confidence ball $C_n$ is constructed that adapts to unknown smoothness and Sobolev-norm of the infinite-dimensional parameter to be estimated. The confidence ball has exact and honest asymptotic coverage over appropriately defined 'self-similar' parameter spaces. It is shown by information-theoretic methods that this 'self-similarity' condition is weakest possible.


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*Keywords:* Adaptation; Confidence sets

## 1. Introduction

Successful statistical methodology in high-dimensional and nonparametric models gives rise, either by construction or implicitly, to statistical procedures that *adapt to unknown properties* of the parameter, such as smoothness or sparsity. It is well-known by now [14,13,3,17,9, 12,1,2,15,6] that such adaptive procedures cannot straightforwardly be used for uncertainty quantification. Particularly, and unlike in the classical parametric situation, adaptive estimators

* Corresponding author.
*E-mail address:* r.nickl@statslab.cam.ac.uk (R. Nickl).

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do *not* automatically suggest valid confidence sets for natural high- or infinite dimensional parameters. Rather, some additional constraints on the parameter space have to be introduced.

In nonparametric models one such constraint that is naturally compatible with the desired adaptation properties has been studied in [10,12,1,18,7]—the term 'self-similarity assumption' has been associated with this condition, for reasons that will become apparent below. Except for [18], the above references have studied such parameter constraints in the '$L^\infty$-setting' of confidence *bands*, pertaining to the uniform-norm as a statistical loss function. The situation in the '$L^2$-setting' – where the risk function is induced by the more common integrated squared loss – is in principle more favourable (see [13,4,17,2,5,19]), and for certain ranges of parameter spaces such 'self-similarity' conditions are simply not necessary. However, as will be explained below, for the most meaningful adaptation problems that range over a full scale of Sobolev spaces with possibly unbounded Sobolev-norm of the function to be estimated, the situation becomes more delicate and 'self-similarity conditions' are relevant again.

In the present article we consider the basic nonparametric sequence space model and provide minimal $\ell^2$-type self-similarity constraints on a Sobolev-parameter space that cannot be improved upon from an information theoretic point of view. We also show that an easy to construct, asymptotically exact, confidence ball based on the idea of unbiased risk estimation performs optimally under such constraints. In contrast to most constructions in the literature, no 'under-smoothing' is necessary, and the confidence set adapts to minimax rate of convergence and radius constant.

The interest in this problem is partly triggered by recent progress on the understanding of the frequentist properties of Bayesian uncertainty quantification methods in [18], where $L^2$-type self-similarity conditions have been employed successfully. Combined with some arguments of [19] our results imply that natural nonparametric Bayes approaches based on Gaussian priors with hierarchical or maximum marginal likelihood empirical Bayes prior specification of the smoothness parameter do *not* achieve the information theoretic limits of uncertainty quantification.

As usual our ideas and techniques carry over from the sequence space model to more common nonparametric regression and density estimation problems, both constructively by virtue of the $L^2 \sim \ell^2$ isometry of the loss functions, and more fundamentally through asymptotic equivalence theory for statistical experiments.

## 2. Main results

Consider observations $Y = (y_k : k \in \mathbb{N})$ in the Gaussian sequence space model

$$y_k = f_k + \frac{1}{\sqrt{n}} g_k, \qquad g_k \overset{i.i.d.}{\sim} N(0,1), \quad k \in \mathbb{N}, \tag{1}$$

and write $\Pr_f$ or $\Pr_f^{(n)}$ for the law of $(y_k : k \in \mathbb{N})$. The symbol $E_f$ or $E_f^{(n)}$ denotes expectation under the law $\Pr_f$. Let us assume that the unknown sequence of interest $f = (f_k) \in \ell^2$ belongs to a Sobolev ball, that is, an ellipsoid in $\ell^2$ of the form

$$S^s(B) = \{f \in \ell^2 : \|f\|_{s,2} \le B\}, \quad s > 0,\ B > 0,$$

where the Sobolev norm is given by

$$\|f\|_{s,2}^2 = \sum_{k=1}^{\infty} f_k^2 k^{2s}.$$

Note that $\|\cdot\|_2 \equiv \|\cdot\|_{0,2}$ is the usual $\ell^2$-norm.