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An Itô calculus for a class of limit processes arising from random walks on the complex plane

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Abstract

Within the framework of the previous paper (Bonaccorsi and Mazzucchi, 2015), we develop a generalized stochastic calculus for processes associated to higher order diffusion operators. Applications to the study of a Cauchy problem, a Feynman–Kac formula and a representation formula for higher derivatives of analytic functions are also given.

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1. Introduction

One of the main instances of the fruitful interplay between analysis and probability is the connection between parabolic equations associated to second-order elliptic operators and the theory of Markov processes. The main consequence of this extensively studied topic is the

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famous *Feynman–Kac formula*, providing a representation of the solution of the heat equation with potential $V \in C_0^\infty(\mathbb{R}^d)$ (the continuous functions vanishing at infinity)

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x), \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d \\ u(0,x) = f(x) \end{cases} \tag{1}$$

in terms of an integral with respect to the distribution of the Wiener process W, the mathematical model of the Brownian motion:

$$u(t,x) = \mathbb{E}\left[e^{-\int_0^t V(W(s)+x)\,\mathrm{d}s} f(W(t)+x)\right]. \tag{2}$$

In fact a probabilistic representation of this form cannot be written in the case of semigroups whose generator does not satisfy the maximum principle. In particular if the Laplacian in Eq. (1) is replaced with a higher order differential operator, i.e. if we consider a Cauchy problem of the form

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = (-1)^{n+1}\Delta^n u(t,x) - V(x)u(t,x), \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ u(0,x) = f(x), \end{cases} \tag{3}$$

with $n \in \mathbb{N}$, $n \geq 2$, then a formula analogous to (2), giving the solution of (3) in terms of the expectation with respect to the measure associated to a Markov process, is lacking and it is not possible to find a stochastic process which plays for the parabolic equation (3) the same role that the Wiener process plays for the heat equation. The problem of how to overcome this limitation has been studied by means of different techniques and two main approaches have been proposed. The first one was introduced by V. Yu. Krylov in 1960 [16] and further developed by K. Hochberg in 1978 [12]. The solution of (3) is constructed in terms of the expectation with respect to a *signed* measure with infinite total variation on a space of paths on the interval $[0, t]$. This approach is related to the theory of *pseudoprocesses*, i.e. processes associated to signed instead of probability measures. It is important to recall that due to the particular conditions necessary for the generalization of the Kolmogorov existence theorem for the limit of a projective system of complex measures (see [27]), in the case of Krylov–Hochberg process, a well defined signed measure on $\mathbb{R}^{[0,t]}$ cannot exist and the “integrals” realizing the Feynman–Kac formula for Eq. (3) are just formal expressions which cannot make sense in the framework of Lebesgue integration theory but are to be meant as limit of a particular approximating sequence. However, even taking into account these technical problems, an analog of the arc-sine law [12,14,17], of the central limit theorem [13,26] and of Itô formula and Ito stochastic calculus [12,22] have been developed for the (finite additive) Krylov–Hochberg signed measure. For an extensive discussion of these problems in the framework of a generalized integration theory on infinite dimensional spaces as well as for a unified view of probabilistic and complex integration see [2,1]. It is worthwhile to mention the work by D. Levin and T. Lyons [19] on rough paths, conjecturing that the above mentioned signed measure could be finite if defined on the quotient space of equivalence classes of paths corresponding to different parametrization of the same path.

A different approach, introduced by T. Funaki [11] for the case where $n = 2$, is based on the construction of a stochastic process (with dependent increments) on the complex plane. Funaki’s process is obtained by composing two independent Brownian motions and has some relations with the *iterated Brownian motion* [8,3]. Furthermore this approach is related with the theory of Bochner subordination [6] and can be applied to partial differential equations