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Stochastic Processes and their Applications 127 (2017) 2864–2899

stochastic processes and their applications

www.elsevier.com/locate/spa

One dimensional random walks killed on a finite set

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Received 7 March 2016; received in revised form 23 December 2016; accepted 6 January 2017
Available online 16 January 2017

Abstract

We study the transition probability, say $p^n_A(x, y)$, of a one-dimensional random walk on the integer lattice killed when entering into a non-empty finite set A. The random walk is assumed to be irreducible and have zero mean and a finite variance σ^2. We show that $p^n_A(x, y)$ behaves like $[g^+_A(x)\widehat{g}^+_A(y) + g^-_A(x)\widehat{g}^-_A(y)](\sigma^2/2n)p^n(y - x)$ uniformly in the regime characterized by the conditions $|x| \vee |y| = O(\sqrt{n})$ and $|x| \wedge |y| = o(\sqrt{n})$ generally if $xy > 0$ and under a mild additional assumption about the walk if $xy < 0$. Here $p^n(y - x)$ is the transition kernel of the random walk (without killing); $g^\pm_A$ are the Green functions for the 'exterior' of A with 'pole at $\pm\infty$' normalized so that $g^\pm_A(x) \sim 2|x|/\sigma^2$ as $x \to \pm\infty$; and $\widehat{g}^\pm_A$ are the corresponding Green functions for the time-reversed walk.

MSC: primary 60G50; secondary 60J45

Keywords: Exterior domain; Transition probability; Escape from a finite set; Hitting probability of a finite set; Potential theory

1. Introduction and main results

This paper concerns the transition probability of a one-dimensional random walk on the integer lattice $\mathbb{Z}$ killed on a finite set A. For random walks on the d-dimensional integer lattice $\mathbb{Z}^d$, $d \geq 1$ killed on a finite set H. Kesten [7] obtained, among others, the asymptotic form of the

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http://dx.doi.org/10.1016/j.spa.2017.01.003
0304-4149/

transition probability under a quite general setting. For the important case of one dimensional random walk with zero mean and finite variance, however, his result is restricted to the special case when A consists of a single point. In this paper we extend it to every finite set A. Our result is stronger than his in another respect: the asymptotic estimate is valid uniformly for space variables within a reasonable range of relevant variables. It is incidentally revealed that according as the third absolute moment of the increment variable is finite or not, the walk killed at the origin exhibits qualitatively different behaviour as the starting and landing positions are taken far from the origin in the opposite directions from each other (see Remark 2 near the end of this section).

Our method of proof is quite different from that of Kesten [7]. In [7] a compactness argument is used as a basic tool. Our proof reflects the behaviour of random walk path. It rests on the results of [12] in which the same problem as the present paper is studied but with $A = \{0\}$ and an asymptotic form of the transition probability valid uniformly for space variables is obtained. The same method is applied in [13] to higher dimensional random walks to obtain a similar strengthening of Kesten's result. For multidimensional Brownian motions the corresponding problem is studied by [3] for space variables restricted to compact sets and by [14] without any restriction as such.

Let $S_n = S_0 + X_1 + \cdots + X_n, n = 1, 2, \ldots$ be a random walk on the one-dimensional integer lattice $\mathbb{Z}$. Here the increments X_j are i.i.d. $\mathbb{Z}$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and the initial state S_0 is an integer left unspecified for now. As usual the law with $S_0 = x$ of the walk (S_n) is denoted by P_x and the corresponding expectation by E_x. Throughout this paper we suppose that the random walk (S_n) is irreducible, namely for every $x \in \mathbb{Z}$, $P_0[S_n = x] > 0$ for some $n > 0$, and that

$$EX = 0 \quad \text{and} \quad 0 < \sigma^2 := EX^2 < \infty. \tag{1.1}$$

Here as well as in what follows X is a random variable having the same law as X_1 and E the expectation w.r.t. P.

Let $p(x) = P[X = x]$ and $p^n(x) = P_0[S_n = x]$ so that for $y \in \mathbb{Z}$,

$$P_x[S_n = y] = p^n(y - x) \quad \text{and} \quad p^0(x) = \delta_{x,0},$$

where $\delta_{x,y}$ equals unity if $x = y$ and zero if $x \neq y$. For a non-empty finite subset A of $\mathbb{Z}$, let $p_A^n(x, y)$ denote the transition probability of the walk S_n killed upon entering A, defined by

$$p_A^n(x, y) = P_x[S_k \notin A \text{ for } 1 \leq k \leq n \text{ and } S_n = y], \quad n = 0, 1, 2, \ldots.$$

Thus $p_A^0(x, y) = \delta_{x,y}$ (even if $y \in A$); and $p_A^n(x, y) = 0$ whenever $y \in A, n \geq 1$.

Let $a(x)$ be the potential function of the walk defined by

$$a(x) = \lim_{n\to\infty} \sum_{k=0}^{n} [p^k(0) - p^k(-x)].$$

It is convenient to bring in

$$a^\dagger(x) := \delta_{x,0} + a(x).$$

The result of Kesten [7] mentioned above implies that for each x and $y \neq 0$, as $n \to \infty$

$$\lim_{n\to\infty} \frac{p_{\{0\}}^n(x, y)}{f_0(n)} = a^\dagger(x)a^\dagger(-y) + \frac{xy}{\sigma^4}, \tag{1.2}$$