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Dynamical moderate deviations for the Curie–Weiss model

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Abstract

We derive moderate deviation principles for the trajectory of the empirical magnetization of the standard Curie–Weiss model via a general analytic approach based on convergence of generators and uniqueness of viscosity solutions for associated Hamilton–Jacobi equations. The moderate asymptotics depend crucially on the phase under consideration.

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1. Introduction

The study of the normalized sum of random variables and its asymptotic behavior plays a central role in probability and statistical mechanics. Whenever the variables are independent and have finite variance, the central limit theorem ensures that the sum with square-root normalization converges to a Gaussian distribution. The generalization of this result to dependent variables

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is particularly interesting in statistical mechanics where the random variables are correlated through an interaction Hamiltonian. Ellis and Newman characterized the distribution of the normalized sum of spins (*empirical magnetization*) for a wide class of mean-field Hamiltonian of Curie–Weiss type [8–10]. They found conditions, in terms of thermodynamic properties, that lead in the infinite volume limit to a Gaussian behavior and those which lead to a higher order exponential probability distribution.

A natural further step was the investigation of large and moderate fluctuations for the magnetization. The large deviation principle is due to Ellis [7]. Moderate deviation properties have been treated by Eichelsbacher and Löwe in [6]. A moderate deviation principle is technically a large deviation principle and consists in a refinement of a (standard or non-standard) central limit theorem, in the sense that it characterizes the exponential decay of the probability of deviations from the average on a smaller scale. In [6], it was shown that the physical phase transition in Curie–Weiss type models is reflected by a radical change in the asymptotic behavior of moderate deviations. Indeed, whereas the rate function is quadratic at non-critical temperatures, it becomes non-quadratic at criticality.

All the results mentioned so far have been derived at equilibrium; on the contrary, we are interested in describing the time evolution of fluctuations, obtaining non-equilibrium properties. Fluctuations for the standard Curie–Weiss model were studied on the level of a path-space large deviation principle by Comets [2] and Kraaij [15] and on the level of a path-space central limit theorem by Collet and Dai Pra in [1]. The purpose of the present paper is to study dynamical moderate deviations to complete the analysis of fluctuations of the empirical magnetization. We apply the generator convergence approach to large deviations by Feng–Kurtz [13] to characterize the most likely behavior for the trajectories of fluctuations around the stationary solution(s) in the various regimes. The moderate asymptotics depend crucially on the phase we are considering. The criticality of the inverse temperature $\beta = 1$ shows up at this level via a sudden change in the speed and rate function of the moderate deviation principle for the magnetization. In particular, our findings indicate that fluctuations are Gaussian-like in the sub- and super-critical regimes, while they are not at the critical point.

Besides, we analyze the deviation behavior when the temperature is size-dependent and is increasing to the critical point. In this case, the rate function inherits features of both the uniqueness and multiple phases: it is the combination of the critical and non-critical rate functions. To conclude, it is worth to mention that our statements are in agreement with the results found in [6].

The outline of the paper is as follows: in Section 2 we formally introduce the Curie–Weiss model and we state our main results. All the proofs, if not immediate, are postponed to Section 3. Appendix is devoted to the derivation of a large deviation principle via solving a Hamilton–Jacobi equation and it is included to make the paper as much self-contained as possible.

2. Model and main results

2.1. Notation and definitions

Before we give our main results, we introduce some notation. We start with the definition of good rate-functions and what it means for random variables to satisfy a large deviation principle.

Definition 2.1. Let $X_1, X_2, \ldots$ be random variables on a Polish space F. Furthermore let $I : F \to [0, \infty]$ and let $\{r(n)\}_{n \geq 1}$ be a sequence of positive numbers such that $r(n) \to \infty$.