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Intrinsic expansions for averaged diffusion processes

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Abstract

We show that the convergence rate of asymptotic expansions for solutions of SDEs is higher in the case of degenerate diffusion compared to the elliptic case, i.e. it is higher when the Brownian motion directly acts only along some directions. In the scalar case, this phenomenon was already observed in Gobet and Miri 2014 using Malliavin calculus techniques. Here, we provide a general and detailed analysis by employing the recent study of intrinsic functional spaces related to hypoelliptic Kolmogorov operators in Pagliarani et al. 2016. Applications to finance are discussed, in the study of path-dependent derivatives (e.g. Asian options) and in models incorporating dependence on past information.

Keywords: Averaged diffusion; Hypoelliptic Kolmogorov operators; Asymptotic expansion; Asian option

1. Introduction

We consider a continuous $\mathbb{R}^d$-valued Feller process $X = (X_t)_{t\in[0,T]}$ defined, according to [16], on the space $\big(\Omega, \mathcal{F}, (\mathcal{F}_s^t)_{0\le t\le s\le T}, (P_{t,x})_{(t,x)\in[0,T]\times D}\big)$ where D is a domain of $\mathbb{R}^d$ and $T > 0$ is fixed. We assume that the infinitesimal generator of X coincides on $[0,T]\times D$ with a

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second order differential operator of the form

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{p_0} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{p_0} a_i(t,x)\partial_{x_i} + \langle Bx, \nabla_x\rangle, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{1.1}$$

where $p_0 \le d$ and $\mathcal{A}$ verifies the following

Assumption 1.1. $A_0 := \big(a_{ij}(t,x)\big)_{i,j=1,\dots,p_0}$ satisfies the non-degeneracy condition

$$\mu M|\xi|^2 < \sum_{i,j=1}^{p_0} a_{ij}(t,x)\xi_i\xi_j < M|\xi|^2, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d,\ \xi\in\mathbb{R}^{p_0}, \tag{1.2}$$

for some positive constants M and μ;

Assumption 1.2. B is a $(d \times d)$-matrix with constant entries satisfying the following structural condition

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & 0 \end{pmatrix} \tag{1.3}$$

where each B_j is a $(p_j \times p_{j-1})$-matrix of rank p_j and

$$p_0 \ge p_1 \ge \cdots \ge p_r \ge 1, \qquad \sum_{j=0}^{r} p_j = d.$$

Example 1.3. Let $X = (S, A)$ where

$$dS_t = \sigma S_t dW_t, \qquad dA_t = S_t dt, \tag{1.4}$$

are the price and average processes in the Black & Scholes model for the pricing of arithmetic Asian options. In this case, we have $d = 2$, $p_0 = 1$ and

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The infinitesimal generator of X is

$$\frac{\sigma^2 x_1^2}{2}\partial_{x_1 x_1} + x_1\partial_{x_2}, \quad (x_1,x_2)\in\mathbb{R}_{>0}\times\mathbb{R}_{>0},$$

and clearly coincides, *on any domain D compactly contained in* $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$, with an operator of the form (1.1), that satisfies Assumptions 1.1 and 1.2. Clearly, more sophisticated multi-dimensional models, including stochastic and local volatility, as well as non-null interest rates, fall within the general framework above.

Assumption 1.2 implies that vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and

$$Y := \langle Bx, \nabla_x\rangle + \partial_t \tag{1.5}$$