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Existence and estimates of moments for Lévy-type processes

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Abstract

In this paper, we establish the existence of moments and moment estimates for Lévy-type processes. We discuss whether the existence of moments is a time dependent distributional property, give sufficient conditions for the existence of moments and prove estimates of fractional moments. Our results apply in particular to SDEs and stable-like processes.

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1. Introduction

For a Lévy process $(X_t)_{t\geq 0}$ and a submultiplicative function $f \geq 0$ it is known

(i) ...that the existence of the generalized moment $\mathbb{E}f(X_t)$ does not depend on time, i.e. $\mathbb{E}f(X_{t_0}) < \infty$ for some $t_0 > 0$ implies $\mathbb{E}f(X_t) < \infty$ for all $t \geq 0$, see e.g. [15, Theorem 25.18].
(ii) ...that the existence of moments be characterized in terms of the Lévy triplet, see e.g. [15, Theorem 25.3].
(iii) ...what the small-time asymptotics of fractional moments $\mathbb{E}(|X_t|^\alpha), \alpha > 0$, looks like, cf. [5,12].

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The first two problems are of fundamental interest; the asymptotics of fractional moments has turned out to be of importance in various parts of probability theory, e.g. to obtain Harnack inequalities [5] or to prove the existence of densities for solutions of stochastic differential equations [7]. Up to now, there is very little known about the answers for the larger class of Lévy-type processes which includes, in particular, stable-like processes, affine processes and solutions of (Lévy-driven) stochastic differential equations. The aim of this work is to extend results which are known for Lévy processes from the Lévy case to Lévy-type processes.

In the last years, heat kernel estimates for Lévy(-type) processes have attracted a lot of attention. Let us point out that the results obtained here have several applications in this area. In a future work, we will show that any rich Lévy-type process $(X_t)_{t\geq 0}$ with triplet $(b(x), Q(x), N(x, dy))$ satisfies the integrated heat kernel estimate

$$\frac{\mathbb{P}^x(|X_t - x| \geq R)}{t} \xrightarrow{t\to 0} N(x, \{y \in \mathbb{R}^d; |y| \geq R\}) \tag{1}$$

for all $R > 0$ such that $N(x, \{y \in \mathbb{R}^d; |y| = R\}) = 0$. Combining this with the statements from Section 4 gives the small-time asymptotics of $t^{-1}\mathbb{E}^x f(X_t)$ for a large class of functions f; the functions need not to be bounded or differentiable. The corresponding results for Lévy processes have been discussed by Jacod [10] and Figueroa-López [6]. As suggested in [6], this gives the possibility to extend the generator of the process to a larger class of functions. Moreover, following a similar approach as Fournier and Printems [7], the estimates of the fractional moments show the existence of (L^2-)densities for Lévy-type processes with Hölder-continuous symbols.

The structure of this paper is as follows. In Section 2, we introduce basic definitions and notation. The problems mentioned above will be answered in Sections 3–5; starting with the question whether the existence of moments is a time dependent distributional property in Section 3, we give sufficient conditions for the existence of moments in Section 4 and finally present estimates of fractional moments in Section 5. In each of these sections, we give a brief overview on known results, state some generalizations and illustrate them with examples.

2. Basic definitions and notation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For a random variable X on $(\Omega, \mathcal{A}, \mathbb{P})$ we denote by $\mathbb{P}_X$ the distribution of X with respect to $\mathbb{P}$. We say that two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ are *comparable* and write $f \asymp g$ if there exists a constant $c > 0$ such that $c^{-1} f(x) \leq g(x) \leq cf(x)$ for all $x \in \mathbb{R}^d$. Moreover, we denote by $\mathcal{B}_b(\mathbb{R}^d)$ the space of all bounded Borel-measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ and by $C_c^2(\mathbb{R}^d)$ the space of functions with compact support which are twice continuously differentiable. For $x \in \mathbb{R}^d$ and $r > 0$ we set $B(x, r) := \{y \in \mathbb{R}^d; |y-x| < r\}$ and $B[x, r] := \{y \in \mathbb{R}^d; |y - x| \leq r\}$. The jth unit vector in $\mathbb{R}^d$ is denoted by e_j and $x \cdot y = \sum_{j=1}^d x_j y_j$ is the Euclidean scalar product. For a function $u : \mathbb{R}^d \to \mathbb{R}$ we denote by $\partial_{x_j}^k u(x)$ the kth order partial derivative with respect to x_j. The *Fourier transform* of an integrable function $u : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\hat{u}(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\,x\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^d.$$

We call a stochastic process $(L_t)_{t\geq 0}$ a (d-dimensional) *Lévy process* if $L_0 = 0$ almost surely, $(L_t)_{t\geq 0}$ has stationary and independent increments for all $s \leq t$ and $t \mapsto L_t(\omega)$ is càdlàg for