Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 127 (2017) 1496–1516

stochastic processes and their applications

www.elsevier.com/locate/spa

Finite dimensional Fokker–Planck equations for continuous time random walk limits

Ofer Busani

Bar Ilan University, Israel

Received 21 March 2016; received in revised form 18 August 2016; accepted 29 August 2016
Available online 22 September 2016

Abstract

Continuous Time Random Walk (CTRW) is a model where particle's jumps in space are coupled with waiting times before each jump. A Continuous Time Random Walk Limit (CTRWL) is obtained by a limit procedure on a CTRW and can be used to model anomalous diffusion. The distribution $p(dx, t)$ of a CTRWL X_t satisfies a Fractional Fokker–Planck Equation (FFPE). Since CTRWLs are usually not Markovian, their one dimensional FFPE is not enough to completely determine them. In this paper we find the FFPEs of the distribution of X_t at multiple times, i.e. the distribution of the random vector $\left(X_{t_1}, \ldots, X_{t_n}\right)$ for $t_1 < \cdots < t_n$ for a large class of CTRWLs. This allows us to define CTRWLs by their finite dimensional FFPEs.

MSC: 60G50; 60F17; 82C31

Keywords: Continuous time random walks limits; Fractional Fokker–Planck equation; Anomalous diffusion

1. Introduction

Continuous Time Random Walk (CTRW) models the movement of a particle in space, where the k'th jump J_k of the particle in space succeeds the k'th waiting time W_k. We let $N_t = \sup\{k : T_k \leq t\}$ where $T_k = \sum_{i=1}^{k} W_i$, if $T_1 > t$ then we set N_t to be 0. N_t is just the number of jumps of the particle up to time t. Then

$$X'_t = \sum_{k=1}^{N_t} J_k,$$

E-mail address: ofer659@gmail.com.

http://dx.doi.org/10.1016/j.spa.2016.08.008
0304-4149/

is the CTRW associated with the time–space jumps $\{(J_k, W_k)\}_{k\in\mathbb{N}}$. Let us now assume that $\{J_k\}$ and $\{W_k\}$ are independent i.i.d. sequences of random variables. In order to model the long time behaviour of the CTRW we write $\left\{\left(J_k^c, W_k^c\right)\right\}_{k\in\mathbb{N}}$ for $c > 0$. Here the purpose of c is to facilitate the convergence of the trajectories of $\left\{\left(J_k^c, W_k^c\right)\right\}_{k\in\mathbb{N}}$ weakly on a proper space. More precisely, we let $\mathcal{D}\left([0,\infty), \mathbb{R}^2\right)$ be the space of cádlág functions $f : [0,\infty) \to \mathbb{R}^2$ equipped with the Skorokhod J_1 topology. We assume that

$$\left(S_u^c, T_u^c\right) = \sum_{k=1}^{\lfloor cu \rfloor} \left(J_k^c, W_k^c\right) \Rightarrow (A_u, D_u) \quad c \to \infty,$$

where $\Rightarrow$ denotes weak convergence of measures with respect to the J_1 topology. We further assume that the processes A_t and D_t are independent Lévy processes and that D_t is a strictly increasing subordinator. Denote by X_t^c the CTRW associated with $\left\{\left(J_k^c, W_k^c\right)\right\}_{k\in\mathbb{N}}$. We then have ([15, Theorem 3.6] and [14, Lemma 2.4.5])

$$X_t^c \Rightarrow X_t = A_{E_t} \quad c \to \infty, \tag{1.1}$$

where $E_t = \inf\{s : D_s > t\}$ is the inverse of D_t and $\Rightarrow$ means weak convergence on $\mathcal{D}([0,\infty), \mathbb{R})$ equipped with the J_1 topology. It is well known that X_t is usually not Markovian, a fact that makes the task of finding basic properties of X_t nontrivial. One such task is finding the finite dimensional distributions (FDDs) of the process X_t, i.e. $P\left(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\right)$. In the physics literature, there is much emphasis put on the FDDs and correlation functions of the Continuous Time Random Walk Limit (CTRWL). Correlation functions are a vital experimental tool for distinguishing CTRWL from other fractional diffusion (such as fractional Brownian motion) [2]. In [11], Meerschaert and Straka used a semi-Markov approach to find the FDDs for a large class of CTRWL. It turns out that the discrete regeneration times of X_t^c converge to a set of points where X_t is renewed. Once we know the next time of regeneration of X_t, we no longer need older observations in order to determine the future behaviour of X_t. More mathematically, denote by $R_t = D_{E_t} - t$ the time left before regeneration of X_t then (X_t, R_t) is a Markov process. One can then use the transition probabilities of (X_t, R_t) along with the Chapman–Kolmogorov Equations in order to find $P\left(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\right)$ for $t_1 < \cdots < t_n$ and $n \in \mathbb{N}$. This method was used in [6] in order to find the FDD of the aged process $X_t^{t_0} = X_t - X_{t_0}$. It is well known [9, Section 4.5] that the one dimensional distribution $p(dx, t) = P(X_t \in dx)$ satisfies a Fractional Fokker–Planck Equation (FFPE). Once again, as X_t is non Markovian the FFPE satisfied by $p(dx, t)$ is not enough to fully describe X_t (as it does when X_t is Markovian). Hence, a dual problem to finding the FDDs is that of finding the finite dimensional FFPEs of the FDDs of X_t. In this paper we obtain the finite dimensional FFPEs for a large class of CTRWL. We use the expression of the FDDs found in [11] along with Fourier–Laplace transform to find the FFPEs of these FDD. This is done first by investigating the multivariable Fourier–Laplace transform on relevant distributions on certain subsets of $\mathbb{R}_+^n$, developing multivariable space–time pseudo-differential operators (PDOs) and applying these results to the expression found in [11]. Results on the finite dimensional FFPEs of CTRWL exist in the literature [3,4,7], however, the methods used there are somewhat limited (cf. Remark 4). For example, these methods can only be used to find the FFPEs of the distribution $h(dx_1, \ldots, dx_n; t_1, \ldots, t_n)$ of the inverse of a subordinator on $x_1 < x_2 < \cdots < x_n$, whereas the distribution's support is $x_1 \le x_2 \le \cdots \le x_n$. Moreover, these methods are ill-suited for coupled CTRWs. Our results generalize prior results to find the FFPE of the inverse subordinator on $x_1 \le x_2 \le \cdots \le x_n$ as well as for the