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Normal approximation and almost sure central limit theorem for non-symmetric Rademacher functionals

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Abstract

In this work, we study the normal approximation and almost sure central limit theorems for some functionals of an independent sequence of Rademacher random variables. In particular, we provide a new chain rule that improves the one derived by Nourdin et al. (2010) and then we deduce the bound on Wasserstein distance for normal approximation using the (discrete) Malliavin–Stein approach. Besides, we are able to give the almost sure central limit theorem for a sequence of random variables inside a fixed Rademacher chaos using the *Ibragimov–Lifshits criterion*.

Keywords: Rademacher functional; Normal approximation; Wasserstein distance; Almost sure central limit theorem; Malliavin–Stein approach

1. Introduction

This work is devoted to the study of discrete Malliavin–Stein approach for two kinds of Rademacher functionals:

(S) $Y_k, k \in \mathbb{N}$ is a sequence of independent identically distributed (*i.i.d*) Rademacher random variables, *i.e.* $\mathbb{P}(Y_1 = -1) = \mathbb{P}(Y_1 = 1) = 1/2$. $F = f(Y_1, Y_2, \dots)$, for some nice function f, is called a (symmetric) Rademacher functional over (Y_k).

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(NS) $X_k, k \in \mathbb{N}$ is a sequence of independent non-symmetric, non-homogeneous Rademacher random variables, that is, $\mathbb{P}(X_k = 1) = p_k$, $\mathbb{P}(X_k = -1) = q_k$ for each $k \in \mathbb{N}$. Here $1 - q_k = p_k \in (0, 1)$ for each $k \in \mathbb{N}$. Of course this sequence reduces to the i.i.d. one when $p_k = q_k = 1/2$ for each k. $G = f(X_1, X_2, \ldots)$, for some nice function f, is called a (non-symmetric) Rademacher functional over (X_k). Usually, we consider the corresponding normalised sequence $(Y_k, k \in \mathbb{N})$ of X_k, that is, $Y_k := (X_k - p_k + q_k) \cdot (2\sqrt{p_k q_k})^{-1}$.

From now on, we write (S) for the symmetric setting, and (NS) for the non-symmetric, non-homogeneous setting.

Now let us explain several terms in the title. Malliavin–Stein method stands for the combination of two powerful tools in probability theory: Paul Malliavin's differential calculus and Charles Stein's method of normal approximation. This intersection originates from the seminal paper [12] by Nourdin and Peccati, who were able to associate a quantitative bound to the remarkable fourth moment theorem established by Nualart and Peccati [15] among many other things. For a comprehensive overview, one can refer to the website [11] and the recent monograph [13].

This method has found its extension to discrete settings: for the Poisson setting, see *e.g.* [16,20]; for the Rademacher setting, the paper [14] by Nourdin, Peccati and Reinert was the first one to carry out the analysis of normal approximation for Rademacher functionals (possibly depending on infinitely many Rademacher variables) in the setting (S), and they were able to get a sufficient condition in terms of contractions for a central limit theorem (CLT) inside a fixed Rademacher chaos $\mathscr{C}_m$ (with $m \geq 2$), see Proposition 3.2 for the precise statement.

In the Rademacher setting, unlike the Gaussian case, one does not have the chain rule like $Df(F) = f'(F)DF$ for $f \in C_b^1(\mathbb{R})$ and Malliavin differentiable random variable F (see [13, Proposition 2.3.7]), while an approximate chain rule (see (2.6)) is derived in [14] and it requires quite much regularity of the function f. As a consequence, the authors of [14] had to use smooth test functions when they applied the Stein's estimation: roughly speaking, for nice centred Rademacher functional F in the setting (S), for $h \in C_b^2(\mathbb{R})$, $Z \sim \mathscr{N}(0, 1)$, one has (see [14, Theorem 3.1])

$$\left|\mathbb{E}\big[h(F) - h(Z)\big]\right| \leq \min\big(4\|h\|_\infty, \|h''\|_\infty\big) \cdot \mathbb{E}\Big[\big|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\big|\Big] + \frac{20}{3}\|h''\|_\infty \mathbb{E}\Big[\big\langle |DL^{-1}F|, |DF|^3\big\rangle_{\mathfrak{H}}\Big], \tag{1.1}$$

where the precise meaning of the above notation will be explained in Section 2.

Krokowski, Reichenbachs and Thäle, carefully using a representation of the discrete Malliavin derivative $Df(F)$ and the fundamental theorem of calculus instead of the approximate chain rule (2.6), were able to deduce the Berry–Esséen bound in [8, Theorem 3.1] and its non-symmetric analogue in [9, Proposition 4.1]: roughly speaking, for nice centred Rademacher functional F in the setting (NS),

$$d_K\big(F, Z\big) := \sup_{x\in\mathbb{R}}\Big|\mathbb{P}(F \leq x) - \mathbb{P}(Z \leq x)\Big| \tag{1.2}$$

$$\leq \mathbb{E}\Big[\big|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\big|\Big] + \frac{\sqrt{2\pi}}{8}\mathbb{E}\Big[\Big\langle \frac{1}{\sqrt{pq}}|DL^{-1}F|, |DF|^2\Big\rangle_{\mathfrak{H}}\Big] \tag{1.3}$$

$$+ \frac{1}{2}\mathbb{E}\Big[\Big\langle |F \cdot DL^{-1}F|, \frac{1}{\sqrt{pq}}|DF|^2\Big\rangle_{\mathfrak{H}}\Big]$$

$$+ \sup_{x\in\mathbb{R}} \mathbb{E}\Big[\Big\langle |DL^{-1}F|, \frac{1}{\sqrt{pq}}(DF) \cdot \mathbb{1}_{(F>x)}\Big\rangle_{\mathfrak{H}}\Big]. \tag{1.4}$$