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Stochastic Processes and their Applications 127 (2017) 3331–3353

stochastic processes and their applications

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# Arbitrage theory for non convex financial market models

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Received 9 November 2015; received in revised form 26 August 2016; accepted 1 January 2017
Available online 10 March 2017

**Abstract**

We propose a unified approach where a security market is described by a liquidation value process. This allows to extend the frictionless models of the classical theory as well as the recent proportional transaction costs models to a larger class of financial markets with transaction costs including non proportional trading costs. The usual tools from convex analysis however become inadequate to characterize the absence of arbitrage opportunities in non-convex financial market models. The natural question is to which extent the results of the classical arbitrage theory are still valid. Our contribution is a first attempt to characterize the absence of arbitrage opportunities in non convex financial market models.


*MSC:* 60G44; G11-G13

*Keywords:* Arbitrage theory; Liquidation value; Transaction costs; European options

## 1. Introduction

In classical financial market models, the main assumption is the absence of friction, i.e. risky assets can be traded at their instantaneous prices without taking into account transaction costs or

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http://dx.doi.org/10.1016/j.spa.2017.01.011

liquidity costs as observed in practice. Arbitrage theory is well developed for such models and one of the main results is the fundamental theorem of asset pricing by Dalang–Morton–Willinger in a discrete-time setting [20, Theorem 2.1.1]. The proof is based on the Kreps–Yan theorem which allows to characterize the condition NA as equivalent to the existence of a risk neutral probability measure under which the price process is a martingale. For frictionless continuous-time models, the main result is that there is no arbitrage in an asymptotic sense if and only if there is an equivalent local martingale measure, [8].

Arbitrage theory for financial market models with transaction costs [11] was initiated by Jouini and Kallal [13] for stock markets defined by bid–ask intervals $[S^b, S^a]$. This is a generalization of frictionless models where $S^b = S^a = S$ contrarily to the real world, e.g. in the presence of order books.

The Schachermayer and Kabanov approaches to define financial markets with proportional transaction costs have been extensively developed and lead to interesting problems due to the tractability of the models, such as hedging and pricing [6,12], optimal consumption [7,28]. In arbitrage theory, this is also a topic of growing interest. One of the main results is the generalization of [13], i.e. the equivalence between absence of arbitrage opportunity and the existence of a martingale evolving in the bid–ask intervals [11,27]. Kabanov provided a unified approach by introducing geometrical financial market models, [20, Chapter 3], where it is possible to exchange any risky asset to another one paying proportional transaction costs. The portfolio positions are vector-valued as they are expressed in physical units contrarily to frictionless models where the liquidation value is a random linear function of the financial position. The main concept is the so-called *solvency cone* of all *solvent positions*, i.e. portfolio positions that can be liquidated without any debt. The usual no arbitrage conditions exclude the possibility for an agent to obtain a solvent terminal portfolio position (resp. a non negative liquidation value) when starting from the zero initial endowment except the zero terminal value. Different versions of the fundamental theorem of asset pricing were obtained in [19,27] for discrete-time models and in [10] for continuous models. Absence of arbitrage opportunities is mainly characterized by the existence of martingales evolving in the positive dual of the solvency cone [20, Theorems 3.2.1 and 3.2.2], see also [17,15,16,18].

In practice, transaction costs may be decomposed into two parts: a brokerage fee and a liquidity cost. Typically, when the trading volume is small, the liquidity cost is negligible while the unit brokerage fee is relatively large with respect to the portfolio value. The unit cost appears to gradually decrease when the trading volume size increases. This can be understood as a kind of order size priority which is observed in the real world. As the brokerage fee is generally a concave function of the trading volume, attention is paid to financial market models with concave transaction costs, see [24,29]. Indeed, when the traded volume is large, transactions may be executed far from the best bid–ask prices, e.g. due to the lack of security supply or demand. In that case, the liquidity costs are not negligible and should be a convex function of the trading volume. Moreover, not only large exchanges make prices executed higher but they might also have a significant lasting effect on future prices.

Modelling market impact and liquidity risk is of growing interest in mathematical finance. In their seminal paper [1], Almgren and Chris suppose that market impact is the result of temporary and permanent impacts, both depending on the trading intensity. Therefore, risky assets are traded through infinitely small orders so that block trades take time to be executed to avoid substantial, if not infinite, transaction costs. On the other hand, Bank and Baum [3] assume that the price process is continually impacted by the cumulative holdings of large traders while Cetin, Jarrow and Protter [5] suppose that it is only temporarily impacted by the instantaneous traded volume.