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Small ball properties and representation results

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Abstract

We show that small ball estimates together with Hölder continuity assumption allow to obtain new representation results in models with long memory. In order to apply these results, we establish small ball probability estimates for Gaussian processes whose incremental variance admits two-sided estimates and the incremental covariance preserves sign. As a result, we obtain small ball estimates for integral transforms of Wiener processes and of fractional Brownian motion with Volterra kernels.

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1. Introduction

One of the most important questions for financial modeling is the question of replication, which loosely can be formulated as follows. Suppose that a continuous time financial market model is driven by a stochastic process $X = \{X_t, t \in [0, 1]\}$ given on some stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathsf{P})$ satisfying usual assumptions. A contingent claim, modeled by an $\mathcal{F}_1$-measurable random variable ξ, is hedgeable, if it admits the representation

$$\xi = \int_0^1 \psi_t dX_t, \tag{1.1}$$

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with some $\mathbb{F}$-adapted (replicating) process ψ. In the case where X is a Wiener process there are two main representation results. The famous Itô representation theorem establishes (1.1) for centered square integrable random variables ξ. Less known is a result of Dudley [5], who proved that every random variable ξ has representation (1.1). There are also a lot of results for martingales or semimartingales, we will not cite them, as this is not our main concern here.

The case where X is not a semimartingale is less studied. The pioneering results were established in [12] for fractional Brownian motion (fBm) B^H with Hurst index $H > 1/2$. The construction used in [12] relies on the Hölder continuity and a small ball estimates for B^H. This fact was later used in [16,17] to extend the results of [12] to a larger class of integrands. In [16], it is also shown that in the case where $X = W + B^H$ is a sum of a Wiener process and an fBm with $H > 1/2$, any random variable has representation (1.1). It is worth to mention also the article [15], where the existence of a continuous integrand ψ is shown in the fBm case.

The main problem with the specific small ball property assumed in the papers [16,17] is that it is hard to verify. As it was mentioned in [9], an upper bound in small ball probability gives lower estimates for metric entropy, which are usually hard to obtain. On the other hand, the assumptions of [16,17] are not optimal for establishing representation results.

The goal of this paper is twofold. First, we investigate precise conditions needed to obtain the representation results and compare them to the small ball estimates. Second, we analyze carefully how to get an upper bound for small ball probability for Gaussian processes with variation distance $\mathbf{E}|X_t - X_s|^2$ satisfying two-sided power bounds, possibly, with different powers. These two steps allow us to establish the representation results for a wide class of processes. This class includes some Gaussian processes X having non-stationary increments, e.g. processes that can be represented as the integrals of smooth Volterra kernels w.r.t. a Wiener process or fBm.

The paper is organized as follows. In Section 2, we prove representation theorems for Hölder continuous processes satisfying small ball property. In Section 3, we establish the small ball estimates for Gaussian processes whose incremental variance satisfies two-sided power estimates and incremental covariance preserves sign. In Section 4, we prove representation results for the Gaussian processes considered in Section 3, and give examples of processes, for which the representation results are in place. The examples include subfractional Brownian motion, bifractional Brownian motion, and integral transforms of Wiener process and fractional Brownian motion with Volterra kernels.

2. Representation theorems for Hölder continuous processes satisfying small ball estimates

This section is concerned with the representation results of the form (1.1). Here we establish general results for processes satisfying Hölder continuity and small ball assumptions.

Consider an adapted process X satisfying the following assumptions, where $C^\theta[0,1]$ denotes the class of Hölder continuous functions of order θ.

(H) Hölder continuity: $X \in C^\theta[0,1]$ a.s. for some $\theta > 1/2$.

(S) Small ball estimate: there exist positive constants λ, μ, K_1, K_2 such that for all $\varepsilon > 0$, $\Delta > 0$, $s \in [0, 1-\Delta]$

$$\mathbf{P}\left\{ \sup_{s \le t \le s+\Delta} |X_t - X_s| \le \varepsilon \right\} \le K_1 \exp\left\{-K_2 \varepsilon^{-\lambda} \Delta^{\mu}\right\}. \tag{2.1}$$

Remark 2.1. In [17], the author establishes existence of representation (1.1) for a centered Gaussian process. The assumptions of [17] are close to be a particular case of (H) and (S).