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On a class of stochastic partial differential equations

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Abstract

This paper concerns the stochastic partial differential equation with multiplicative noise $\frac{\partial u}{\partial t} = \mathcal{L}u + u\dot{W}$, where $\mathcal{L}$ is the generator of a symmetric Lévy process X, $\dot{W}$ is a Gaussian noise and $u\dot{W}$ is understood both in the senses of Stratonovich and Skorohod. The Feynman–Kac type of representations for the solutions and the moments of the solutions are obtained, and the Hölder continuity of the solutions is also studied. As a byproduct, when $\gamma(x)$ is a nonnegative and nonnegative-definite function, a sufficient and necessary condition for $\int_0^t \int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds$ to be exponentially integrable is obtained.

Keywords: Stochastic partial differential equation; Gaussian noise; Feynman–Kac formula; Malliavin calculus

1. Introduction

In [37], Walsh developed the theory of stochastic integrals with respect to martingale measures and used it to study the stochastic partial differential equations (SPDEs) driven by space–time Gaussian white noise. Dalang in his seminal paper [13] extended the definition of Walsh's stochastic integral and applied it to solve SPDEs with Gaussian noise white in time and homogeneously colored in space (white-colored noise). Recently, the theories on SPDEs with white-colored noise have been extensively developed, and one can refer to, for instance, [11,15, 14,30,35] and the references therein. For the SPDEs with white-colored noise, the methods used in the above-mentioned literature relies on the martingale structure of the noise, and hence cannot

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be applied to the case when the noise is colored in time. On the other hand, SPDEs driven by a Gaussian noise which is colored in time and (possibly) colored in space have attracted more and more attention.

In the present article, we consider the following SPDE in $\mathbb{R}^d$,

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u + u\dot{W}, & t \geq 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

In the above equation, $\mathcal{L}$ is the generator of a Lévy process $\{X_t, t \geq 0\}$, $u_0(x)$ is a continuous and bounded function, and the noise $\dot{W}$ is a (generalized) Gaussian random field independent of X with the covariance function given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = |t-s|^{-\beta_0}\gamma(x-y), \tag{1.2}$$

where $\beta_0 \in (0,1)$ and γ is a symmetric, nonnegative and nonnegative-definite (generalized) function. The product $u\dot{W}$ in (1.1) is understood either in the *Stratonovich* sense or in the *Skorohod* sense. Throughout the paper, we assume that X is a symmetric Lévy process with characteristic exponent $\Psi(\xi)$, i.e., $\mathbb{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$. Note that the symmetry implies that $\Psi(\xi)$ is a real-valued nonnegative function. Furthermore, we assume that X has transition functions denoted by $q_t(x)$, which also entails that $\lim_{|\xi|\to\infty}\Psi(\xi) = \infty$ by Riemann–Lebesgue lemma.

When $\mathcal{L} = \frac{1}{2}\Delta$ where Δ is the Laplacian operator, and $\dot{W}$ is colored in time and white in space, Hu and Nualart [26] investigated the conditions to obtain a unique mild solution for (1.1) in the Skorohod sense, and obtained the Feynman–Kac formula for the moments of the solution. When $\mathcal{L} = \frac{1}{2}\Delta$, and $\dot{W}$ is a fractional white noise with Hurst parameters $H_0 \in (\frac{1}{2}, 1)$ in time and $(H_1, \ldots, H_d) \in (\frac{1}{2}, 1)^d$ in space, i.e., $\beta_0 = 2 - 2H_0$ and $\gamma(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$, Hu et al. [28] obtained a Feynman–Kac formula for a weak solution under the condition $2H_0 + \sum_{i=1}^d H_i > d + 1$ for the SPDE in the Stratonovich sense. This result was extended to the case $\mathcal{L} = -(-\Delta)^{\alpha/2}$ in Chen et al. [9]. A recent paper [24] by Hu et al. studied (1.1) in both senses when $\mathcal{L} = \frac{1}{2}\Delta$ and $\dot{W}$ is a general Gaussian noise, obtained the Feynman–Kac formulas for the solutions and the moments of the solutions, and investigated Hölder continuity of the Feynman–Kac functional and the intermittency of the solutions.

There has been fruitful literature on (1.1) in the sense of Skorohod, especially when $\dot{W}$ is white in time. For instance, when $\mathcal{L} = \frac{1}{2}\Delta$, (1.1) is the well-known *parabolic Anderson model* [1] and has been extensively investigated in, for example, [6,7,33]. Foondun and Khoshnevisan [18, 19] studied the general nonlinear SPDEs. For SPDE (1.1) with space–time colored noise, the intermittency property of the solution was investigated in [5,10] when $\mathcal{L} = \frac{1}{2}\Delta$, and in [4] when $\mathcal{L} = -(-\Delta)^{\alpha/2}$.

The main purpose of the current paper is to study (1.1) in both senses of Stratonovich and Skorohod under the assumptions Hypothesis I in Section 3 and Hypothesis II in Section 5.1 respectively. Under Hypothesis I, we will obtain Feynman–Kac type of representations for a mild solution to (1.1) in the Stratonovich sense and for the moments of the solution (Theorems 4.6 and 4.7). Under Hypothesis II, we will show that the mild solution to (1.1) in the Skorohod sense exists uniquely, and obtain the Feynman–Kac formula for the moments of the solution (Theorems 5.3 and 5.5). Furthermore, under stronger conditions, we can get Hölder continuity of the solutions in both senses (Theorems 4.11 and 5.9). As a byproduct, we show that Hypothesis I