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# A Glivenko–Cantelli theorem for almost additive functions on lattices

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**Abstract**

We develop a Glivenko–Cantelli theory for monotone, almost additive functions of i.i.d. sequences of random variables indexed by $\mathbb{Z}^d$. Under certain conditions on the random sequence, short range correlations are allowed as well. We have an explicit error estimate, consisting of a probabilistic and a geometric part. We apply the results to yield uniform convergence for several quantities arising naturally in statistical physics.


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*Keywords:* Glivenko–Cantelli theory; Uniform convergence; Empirical measures; Large deviations; Statistical mechanics

## 1. Introduction

The classical Glivenko–Cantelli theorem states that the empirical cumulative distribution functions of an increasing set of independent and identically distributed random variables converge *uniformly* to the cumulative population distribution function almost surely. Due to its importance to applications, e.g. statistical learning theory, the Glivenko–Cantelli theorem

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is also called the "fundamental theorem of statistics". The theorem has initiated the study of so-called Glivenko–Cantelli classes as they feature, for instance, in the Vapnik–Chervonenkis theory [24]. Generalizations of the fundamental theorem rewrite the uniform convergence with respect to the real variable as convergence of a *supremum over a family (of sets or functions)* and widen the family over which the supremum is taken, making the statement "more uniform". However, there are limits to this uniformization: For instance, if the original distribution is continuous, there is no convergence if the supremum is taken w.r.t. the family of finite subsets of the reals. Thus, a balance has to be found between the class over which the supremum is taken and the distribution of the random variables, the details of which are often dictated by the application in mind. Another important extension are multivariate Glivenko–Cantelli theorems, where the i.i.d. random variables are generalized to i.i.d. random vectors with possibly dependent coordinates. Such results have been obtained e.g. in [18,22,2,30]. In contrast to the classical one-dimensional Glivenko–Cantelli theorem, where no assumptions on the underlying distribution is necessary, in the higher dimensional case, one has to exclude certain singular continuous measures, cf. Theorem 5.3. The multidimensional version of the Portmanteau theorem provides a hint why such conditions are necessary. We apply these results in Section 5.

To avoid confusion, let us stress that uniform convergence in the classical Glivenko–Cantelli Theorem and in our result involves discontinuous functions, so it is quite different to uniform convergence of differentiable functions, as it is encountered e.g. with power series.

In many models of statistical physics one shows that certain random quantities are self-averaging, i.e. possess a well defined non-random thermodynamic limit. This is not only true for random operators of Schrödinger type, cf. e.g. [21,16,27], but also for spin systems, cf. e.g. [5,6,28,29,1]. Note however that the latter papers, studying the free energy (and derived quantities), heavily use specific properties of the exponential function (entering the free energy) like convexity and smoothness. We lack these properties in the Glivenko–Cantelli setting and are thus dealing with a completely different situation. The geometric ingredients of the proof of the thermodynamic limit can be traced back to papers by Van Hove [23] and Følner [4]. This is why the exhaustion sets used in the thermodynamic limit are associated with their names.

While standard statistical problems concern i.i.d. samples, an independence assumption quickly appears unnatural in statistical physics. Neighboring entities in solid state models (such as atoms or spins) are unlikely to not influence each other. In order to treat physically relevant scenarios one introduces a geometry to encode location and adjacency relations between the random variables, which in turn are used to allow dependencies between close random variables. In the present paper we choose $\mathbb{Z}^d$ as our model of physical space, although our methods should apply to amenable groups as well, at least with an additional monotile condition. The focus on $\mathbb{Z}^d$ allows us to avoid technicalities of amenable groups with monotiles and can thus present our results in a simpler, more transparent manner. Furthermore, we can achieve more explicit error bounds due to the simple geometry of $\mathbb{Z}^d$.

Our main result is Theorem 2.6, which is a Glivenko–Cantelli type theorem for a class of monotone, almost additive functions and suitable distributions of the random variables, allowing spatial dependencies. Our precise hypotheses are spelled out in Assumption 2.1 and Definition 2.3. The theorem can be interpreted as a multi-dimensional ergodic theorem with values in the Banach space of right continuous and bounded functions with sup-norm, i.e. a uniform convergence result. Under slightly strengthened assumptions we obtain an explicit error term for the convergence, which is a sum of a geometric and a probabilistic part, cf. Theorem 2.8. While earlier Banach space valued ergodic theorems, e.g. [10,11], have been restricted to a finite set of colors, we are able to treat the real-valued case. To do this, we have