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An integral representation of dilatively stable processes with independent increments☆

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Abstract

Dilative stability generalizes the property of selfsimilarity for infinitely divisible stochastic processes by introducing an additional scaling in the convolution exponent. Inspired by results of Iglói (2008), we will show how dilatively stable processes with independent increments can be represented by integrals with respect to time-changed Lévy processes. Via a Lamperti-type transformation these representations are shown to be closely connected to translatively stable processes of Ornstein–Uhlenbeck-type, where translative stability generalizes the notion of stationarity. The presented results complement corresponding representations for selfsimilar processes with independent increments known from the literature.

MSC: primary 60G51; secondary 60G18; 60H05

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1. Introduction

Many processes in physics and other sciences show certain space–time scaling properties for which the class of self-similar processes provides a natural tool in stochastic modeling. For

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infinitely divisible processes Iglói [6] introduced a more general scaling property called dilative stability with an additional scaling in the convolution exponent. We denote by $\Psi^X_{t_1,\dots,t_k}$ the *log-characteristic function* or the *Lévy exponent* of $(X_{t_1},\dots,X_{t_k})$ for an infinitely divisible process $X=(X_t)_{t\in\mathbb{T}}$, where $\mathbb{T}$ is either $\mathbb{R}$ or $\mathbb{R}_+=[0,\infty)$ and $t_1,\dots,t_k\in\mathbb{T}$, i.e. $\Psi^X_{t_1,\dots,t_k}:\mathbb{R}^k\to\mathbb{C}$ is the unique continuous function with $\Psi^X_{t_1,\dots,t_k}(0,\dots,0)=0$ and

$$\mathbb{E}\left[\exp\Big(i\sum_{j=1}^k\theta_jX_{t_j}\Big)\right]=\exp\left(\Psi^X_{t_1,\dots,t_k}(\theta_1,\dots,\theta_k)\right)$$

for all $\theta_1,\dots,\theta_k\in\mathbb{R}$. Following [1], the infinitely divisible process X is called (α,δ)*-dilatively stable* for some parameters $\alpha,\delta\in\mathbb{R}$ if

$$\Psi^X_{Tt_1,\dots,Tt_k}(\theta_1,\dots,\theta_k)=T^{\delta}\,\Psi^X_{t_1,\dots,t_k}\left(T^{\alpha-\frac{\delta}{2}}\theta_1,\dots,T^{\alpha-\frac{\delta}{2}}\theta_k\right)\tag{1}$$

holds for all $T>0$, $k\in\mathbb{N}$, $t_1,\dots,t_k\in\mathbb{T}$ and $\theta_1,\dots,\theta_k\in\mathbb{R}$. It is immediately clear that for $\delta=0$ and $\alpha>0$ an (α,δ)-dilatively stable process is α-selfsimilar. We remark that the original definition of Iglói [6] is more restrictive (e.g., the process is assumed to be non-Gaussian and to possess moments of arbitrary order) but we use the more general approach from [1]. The class of dilatively stable processes contains some interesting classes of processes that are not selfsimilar, see [6,1] for details. In particular, additionally assuming weak right-continuity of the infinitely divisible process X, dilative stability of X is equivalent to the notion of aggregate-similarity introduced by Kaj [11], see Proposition 1.5 in [1]. From this point of view, dilatively stable processes naturally appear as the class of limit processes in certain aggregation models as shown in Theorem 3.1 of [12]. Examples of dilatively stable limit processes in aggregation schemes appear in [11,19], see Section 3 in [1] for a detailed analysis.

In this paper we will restrict our considerations to *additive processes* $(X_t)_{t\in\mathbb{T}}$ which are defined as in [21] by the following conditions:

(i) The process has *independent increments*, i.e. for any $t_0<t_1<\cdots<t_n$ in $\mathbb{T}$ the random variables $X_{t_0},X_{t_1}-X_{t_0},X_{t_2}-X_{t_1},\dots,X_{t_n}-X_{t_{n-1}}$ are independent.
(ii) The process is *stochastically continuous*, i.e. $P\{|X_s-X_t|>\varepsilon\}\to0$ as $s\to t\in\mathbb{T}$ for any $\varepsilon>0$.
(iii) The process has *càdlàg paths*, i.e. almost surely the mapping $t\mapsto X_t$ is right-continuous with left limits.
(iv) $X_0=0$ almost surely.

It is well known that additive selfsimilar processes are closely connected to selfdecomposable random variables and thus can be represented as integrals with respect to a Lévy process; cf. Wolfe [23,22], Jurek and Vervaat [8,10] and Sato [20]. In order that the random integrals do properly exist, the Lévy process necessarily must have a finite logarithmic moment. Certain extensions of the integral representation for additive operator-selfsimilar and semi-selfsimilar processes are given in [7,2,15], respectively. Further, the Lamperti transform [14] gives a well known correspondence between selfsimilar processes and stationary processes. The latter are stationary Ornstein–Uhlenbeck (OU) processes in case of additive selfsimilar processes and the integral representation of an additive selfsimilar process is directly related to the integral representation of the corresponding OU-process.

Our aim is to generalize the above mentioned integral representations and connections for the larger class of additive dilatively stable processes in Section 2. As already laid out in sections 2.5