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Multi-class oscillating systems of interacting neurons

Susanne Ditlevsen[a,*], Eva Löcherbach[b]

[a] *Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen, Denmark*

[b] *Université de Cergy-Pontoise, AGM UMR-CNRS 8088, 2 avenue Adolphe Chauvin, F-95302, Cergy-Pontoise Cedex, France*

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Abstract

We consider multi-class systems of interacting nonlinear Hawkes processes modeling several large families of neurons and study their mean field limits. As the total number of neurons goes to infinity we prove that the evolution within each class can be described by a nonlinear limit differential equation driven by a Poisson random measure, and state associated central limit theorems. We study situations in which the limit system exhibits oscillatory behavior, and relate the results to certain piecewise deterministic Markov processes and their diffusion approximations.

MSC: 60G55; 60K35

Keywords: Multivariate nonlinear Hawkes processes; Mean-field approximations; Piecewise deterministic Markov processes; Multi-class systems; Oscillations; Diffusion approximation

1. Introduction

Biological rhythms are ubiquitous in living organisms. The brain controls and helps maintain the internal clock for many of these rhythms, and fundamental questions are how they arise and what is their purpose. Many examples of such biological oscillators can be found in the classical

* Corresponding author.
E-mail address: susanne@math.ku.dk (S. Ditlevsen).

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book by Glass and Mackey (1988) [17]. The motivation for this paper comes from the rhythmic scratch like network activity in the turtle, induced by a mechanical stimulus, and recorded and analyzed by Berg and co-workers [3,5,4,27]. Oscillations in a spinal motoneuron are initiated by the sensory input, and continue by some internal mechanisms for some time after the stimulus is terminated. While mechanisms of rapid processing are well documented in sensory systems, rhythm-generating motor circuits in the spinal cord are poorly understood. The activation leads to an intense synaptic bombardment of both excitatory and inhibitory input, and it is of interest to characterize such network activity, and to build models which can generate self-sustained oscillations.

The aim of this paper is to present a microscopic model describing a large network of interacting neurons which can generate oscillations. The activity of each neuron is represented by a point process, namely, the successive times at which the neuron emits an action potential or a so-called spike. A realization of this point process is called a spike train. It is commonly admitted that the spiking intensity of a neuron, i.e., the infinitesimal probability of emitting an action potential during the next time unit, depends on the past history of the neuron and it is affected by the activity of other neurons in the network. Neurons interact mostly through chemical synapses, where a spike of a pre-synaptic neuron leads to an increase if the synapse is excitatory, or a decrease if the synapse is inhibitory, of the membrane potential of the post-synaptic neuron, possibly after some delay. In neurophysiological terms this is called synaptic integration. When the membrane potential reaches a certain upper threshold, the neuron fires a spike. Thus, excitatory inputs from the neurons in the network increase the firing intensity, and inhibitory inputs decrease it. Hawkes processes provide good models of this synaptic integration phenomenon by the structure of their intensity processes, see (1.1). We refer to Chevallier et al. (2015) [8], Chornoboy et al. (1988) [9], Hansen et al. (2015) [20] and to Reynaud-Bouret et al. (2014) [32] for the use of Hawkes processes in neuronal modeling. For an overview of point processes used as stochastic models for interacting neurons both in discrete and in continuous time and related issues, see also Galves and Löcherbach (2016) [16].

In this paper, we study oscillatory systems of interacting Hawkes processes representing the time occurrences of action potentials of neurons. The system consists of several large populations of neurons. Each population might represent a different functional group of neurons, for example different hierarchical layers in the visual cortex, such as V1 to V4, or the populations can be pools of excitatory and inhibitory neurons in a network. Each neuron is characterized by its spike train, and the whole system is described by multivariate counting processes $Z^N_{k,i}(t)$, $t \geq 0$. Here, $Z^N_{k,i}(t)$ represents the number of spikes of the ith neuron belonging to the kth population, during the time interval $[0, t]$. The number of classes n is fixed, and each class $k = 1, \dots, n$ consists of N_k neurons. The total number of neurons is therefore $N = N_1 + \cdots + N_n$.

Under suitable assumptions, the sequence of counting processes $(Z^N_{k,i})_{1\leq k\leq n, 1\leq i\leq N_k}$ is characterized by its intensity processes $(\lambda^N_{k,i}(t))$ defined through the relation

$$\mathbb{P}(Z^N_{k,i} \text{ has a jump in }]t, t + dt] | \mathcal{F}_t) = \lambda^N_{k,i}(t)dt,$$

where $\mathcal{F}_t = \sigma(Z^N_{k,i}(s),\ s \leq t, 1 \leq k \leq n, 1 \leq i \leq N_k)$. We consider a mean-field framework where $\lambda^N_{k,i}(t)$ is given by

$$\lambda^N_{k,i}(t) = f_k\left(\sum_{l=1}^{n} \frac{1}{N_l} \sum_{1\leq j\leq N_l} \int_{]0,t[} h_{kl}(t-s)\, dZ^N_{l,j}(s)\right). \tag{1.1}$$