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A central limit theorem for the Euler integral of a Gaussian random field

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Abstract

Euler integrals of deterministic functions have recently been shown to have a wide variety of possible applications, including signal processing, data aggregation and network sensing. Adding random noise to these scenarios, as is natural in the majority of applications, leads to a need for statistical analysis, the first step of which requires asymptotic distribution results for estimators. The first such result is provided in this paper, as a central limit theorem for the Euler integral of pure, Gaussian, noise fields.

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Keywords: Random field; Euler integral; Gaussian processes; Central limit theorem

1. Introduction

The Euler characteristic $\chi(A)$ of a nice set A is perhaps the oldest, and most fundamental, of its topological invariants. For a compact $A \subset \mathbb{R}^1$, the Euler characteristic is merely the number of its connected components (each one of which will be an interval, possibly containing only a single point). For $A \subset \mathbb{R}^2$, $\chi(A)$ becomes the number of connected components minus the number of holes, while in three dimensions $\chi(A)$ can be written as the alternating sum of the

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numbers of components, handles and hollows. Similar (and, of course, more precise) definitions as alternating sums of Betti numbers, numbers of facets of simplices of differing dimension (when A is triangulisable) or as indices of critical points when a Morse theoretic setting is appropriate, extend the Euler characteristic to a wide variety of sets in arbitrary dimensions.

However, more important for us is that the Euler characteristic is also a valuation, which means that, when all terms are defined. we have the additivity property,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B). \tag{1.1}$$

Given additivity, it is natural to attempt to use χ to define an integral on a suitable family of functions, and, indeed, to a large extent this can be done. The resulting theory is known as Euler integration.

1.1. Euler integration

Although in many ways Euler integration has its roots in classical Integral Geometry, a more complete and modern theory began to evolve in the 1970's. More importantly for us, however, is that it has experienced a rapid development in the past decade from both applied and theoretical aspects, providing for some elegant and novel results. We shall not attempt to survey these here, since the recent papers of Baryshnikov and Ghrist [4] and Curry et al. [7] provide excellent and broad expositions. Rather, we shall go directly to two definitions.

Definition 1.1. Let $M \subset \mathbb{R}^n$ be compact, with finite Euler characteristic. Then a continuous function $f : M \to \mathbb{R}$ is called tame if the homotopy types of $f^{-1}((-\infty, u])$ and $f^{-1}([u, \infty))$ change only finitely many times as u varies over $\mathbb{R}$, and the Euler characteristic of each set is finite.

Definition 1.2. If $f : M \to \mathbb{R}$ is tame, then the upper Euler integral of f over M is defined by

$$\int_M f \lceil d\chi \rceil \triangleq \int_{u=0}^{\infty} [\chi(f > u) - \chi(f \le -u)]\, du, \tag{1.2}$$

where

$$\chi(f \le u) \triangleq \chi(f^{-1}((-\infty, u])),$$

and

$$\chi(f > u) \triangleq \chi(M) - \chi(f \le u).$$

Reading in between the lines that do *not* appear in the above definition, one would guess that there is also a lower Euler integral (there is!) and that there has to be a more direct way to define an integral that follows from the additivity of (1.1). In fact, this is also true, and, as a result, the Euler integral shares many common properties with the classical theories of integration. However, it is somewhat more delicate, since although (1.1) extends to a finite inclusion–exclusion form, it does not typically extend to the countably infinite case needed for a standard measure based theory of integration. The definition that we have chosen above avoids these issues, and in taking it we follow the lead of Bobrowski and Borman [6] who, by taking (1.2) as a definition rather than a property, save often irritating but unimportant (for our needs) technicalities.