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Stochastic Processes and their Applications 126 (2016) 3310–3330

stochastic processes and their applications

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Extreme eigenvalues of sparse, heavy tailed random matrices

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Received 4 August 2015; received in revised form 24 April 2016; accepted 26 April 2016
Available online 1 June 2016

Abstract

We study the statistics of the largest eigenvalues of $p \times p$ sample covariance matrices $\Sigma_{p,n} = M_{p,n} M_{p,n}^*$ when the entries of the $p \times n$ matrix $M_{p,n}$ are sparse and have a distribution with tail $t^{-\alpha}$, $\alpha > 0$. On average the number of nonzero entries of $M_{p,n}$ is of order $n^{\mu+1}$, $0 \leq \mu \leq 1$. We prove that in the large n limit, the largest eigenvalues are Poissonian if $\alpha < 2(1 + \mu^{-1})$ and converge to a constant in the case $\alpha > 2(1 + \mu^{-1})$. We also extend the results of Benaych-Georges and Péché (2014) in the Hermitian case, removing restrictions on the number of nonzero entries of the matrix.

MSC: primary 60K35; 82B43

Keywords: Random matrices; Heavy tail; Sparse; Eigenvalue distribution

1. Introduction

We study the statistics of the largest eigenvalues of sample covariance matrices of the form $\Sigma = MM^*$ when the entries of M are heavy tailed and sparse. Let x be a complex-valued random variable. We say x has a heavy tailed distribution with parameter α if the (two-sided) tail probability

$$G_\alpha(t) := \mathbb{P}(|x| > t) = L(t)t^{-\alpha}, \quad t > 0$$

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http://dx.doi.org/10.1016/j.spa.2016.04.029
0304-4149/

where $\alpha > 0$ and L is a slowly varying function, i.e.,

$$\lim_{t\to\infty} \frac{L(st)}{L(t)} = 1, \quad \forall s > 0.$$

For each $n \geq 1$, let $y = y(n)$ be a Bernoulli random variable, independent of x, with $\mathbb{P}(y = 1) = n^{\mu-1} = 1 - \mathbb{P}(y = 0)$, where $0 \leq \mu \leq 1$ is a constant. The ensemble of random sample covariance matrices that we study here is defined as follows. For each $n \geq 1$, let $p = p(n) \in \mathbb{Z}_+$ be a function of n such that

$$p/n \to \rho, \quad 0 < \rho \leq 1,$$

as $n \to \infty$. Let $A_{p,n} = [a_{ij}]_{i,j=1}^{p,n}$ and $B_{p,n} = [b_{ij}]_{i,j=1}^{p,n}$ be $p \times n$ random matrices whose entries are i.i.d. copies of x and y, respectively. Form the $p \times n$ sparse matrix $M_{p,n} = A_{p,n} \cdot B_{p,n} = [m_{ij}]_{i,j=1}^{p,n}$ by setting $m_{ij} = a_{ij} b_{ij}$. Then

$$\Sigma_{p,n} := M_{p,n} M_{p,n}^*$$

is the $p \times p$ heavy tailed random sample covariance matrix with parameters α and μ. Note that $\Sigma_{p,n}$ is positive semi-definite so all its eigenvalues are non-negative.

The extreme eigenvalues of $\Sigma_{p,n}$ are the main subject of this paper. We will see that, depending on the tail exponent α and the sparsity exponent μ, when properly rescaled, the top eigenvalues will either converge to the points of a Poisson point process or to the right edge of the Marchenko–Pastur law.

To put our theorems in context, we briefly review past results. The study of extreme eigenvalues of heavy tailed random matrices started with the work of Soshnikov. In [26], he proved that if $0 < \alpha < 2$, the asymptotic behavior of the top eigenvalues of a heavy tailed Hermitian matrix is determined by the behavior of the largest entries of the matrix, i.e., the point process of the largest eigenvalues (properly normalized) converges to a Poisson point process, as in the usual extreme value theory for i.i.d. random variables. This result was extended to sample covariance matrices and for all values of $\alpha \in (0, 4)$ in the work of Auffinger, Ben Arous and Péché [2]. The upper bound on the tail exponent α is optimal as for i.i.d. entries with finite fourth moment, the largest eigenvalues converge to the right edge of the bulk distribution and have Tracy–Widom fluctuations [4,5,16,28]. Eigenvector localization and delocalization were studied in [6]. In the physics literature, many of these results were predicted in the seminal paper of Bouchaud and Cizeau [13].

The largest eigenvalues of sparse Hermitian random matrices with bounded moments were investigated by Benaych-Georges and Péché [7] under the assumptions of at least $\omega(\log n)$ nonzero entries in each row. They extended the results of [15,24], establishing the convergence of the largest eigenvalue to the edge and also obtained results on localization/delocalization of eigenvectors. For bulk statistics in the sparse setting, readers are invited to see Erdős, Knowles, Yau, and Yin [14] and the references therein.

In [8], Benaych-Georges and Péché considered a class of $n \times n$ Hermitian, heavy tailed, sparse matrices. In their work, the authors looked at matrices, where in $n - o(n)$ rows, the number of nonzero entries was asymptotically equal to n^{μ} for $\mu \in (0, 1]$. For the remaining $o(n)$ rows, the number of nonzero entries was no more than n^{μ}. This assumption is well-suited to treat the case of heavy-tailed band matrices. In the last section, we will extend the work of [8] by removing all restrictions on the number of nonzero entries in each row, allowing, for instance, the sparsity to come from the adjacency matrix of an Erdős–Rényi random graph.