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Ergodic theory of the symmetric inclusion process

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Abstract

We prove the existence of a successful coupling for n particles in the symmetric inclusion process. As a consequence we characterise the ergodic measures with finite moments, and obtain sufficient conditions for a measure to converge in the course of time to an invariant product measure.

Keywords: Interacting particle systems; Simple inclusion process; Invariant measures; Coupling

1. Introduction

In [12, Chapter VIII], a rather complete ergodic theory is given for the symmetric exclusion process (SEP). In particular, the only extremal invariant measures for the SEP are Bernoulli measures with constant density. This complete characterisation of the set of invariant measures is quite exceptional and follows from the fact that the SEP is self-dual. As a consequence, invariant measures can be related to *bounded harmonic functions for the finite SEP*. Then, by the construction of a successful coupling of the SEP with a finite number of particles, it is shown that all bounded harmonic functions are constant, i.e. depend only on the number of particles. From this in turn, one can conclude that all invariant measures for the SEP are permutation

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invariant, from which one derives by the De Finetti theorem that they are convex combinations of Bernoulli measures. In [6,7] an attractive version (in the sense of having an attractive interaction between the particles) of the SEP is introduced and called the *simple inclusion process* (SIP). In the SIP, particles perform nearest-neighbour jumps according to a simple symmetric random walk and interact by "inclusion jumps", where pairs of neighbouring particles jump to the same site at rate 1. This analogy between SIP (attractive interaction) and SEP (repulsive interaction) becomes even more apparent in [7], where it is shown that the SIP satisfies the analogue of Liggett's comparison inequality [12, Chapter VIII, Proposition I.7] for the evolution of positive definite symmetric functions. The expectation at time $t > 0$ of such a function in the course of the evolution of n SIP-particles is *larger than in the course of the evolution of n independent random walkers*. In particular this implies that a certain class of product measures is mapped by the evolution under the SIP to measures with positive correlations (as opposed to negative correlations in the SEP).

In this paper, we want to investigate as much as possible the invariant measures of the SIP, i.e., understand its ergodic measures and their attractors. Since the number of particles is unbounded and we want make use of self-duality, we will have to restrict to a set of measures with all moments finite. The main problem is then to construct a successful coupling for two sets of n SIP-particles initially at different locations. This coupling seems possible thanks to the fact that as long as SIP-particles do not collide, i.e., are not at neighbouring positions, they behave as independent random walkers and these can be coupled by the coordinate-wise Ornstein coupling in any dimension. The idea of the coupling of SIP-particles comes from [4] combined with [13]: in [13], it is shown that inclusion particles and independent random walkers can be coupled in such a way that at time t they are $o(\sqrt{t})$ apart; the period of time $[0, (1-\delta)t]$ in which coupling according to [13] is used (stage 1) is then followed by a period of time $[(1-\delta)t, t]$ (stage 2) during which the coordinate-wise Ornstein coupling of independent random walkers is used *both for the independent walkers as well as for the SIP particles*. The only problem for this coupling to be successful is to estimate the probability of being coupled before a collision occurs. One can understand however that such a collision event is highly improbable (as $t \to \infty$), as after a long time, the independent random walkers are much farther apart ($O(\sqrt{t})$) than the distance between the independent random walkers and their inclusion partners ($o(\sqrt{t})$). Once one has the successful coupling of SIP-particles, and as a consequence results on the structure of the invariant measures of the SIP, all these results can be transferred without effort to corresponding results for interacting diffusion processes to which the SIP is a dual process such as the Brownian Energy Process (BEP) and the Brownian Momentum Process (BMP).

For particle systems with unbounded occupation numbers, in general only very little information is available on the structure of the set of invariant measures. See [1] for a set of general results under Lipschitz conditions on the jumps rates. In this work, both the symmetry and the self-duality property are crucial. Indeed, for an asymmetric version of the SIP based on the Lie-algebraic construction, recently introduced in [3], despite self-duality, there is no characterisation of invariant measures, and the translation invariant ones are known not to be product. On the other side, the naive asymmetric version of the SIP (putting different factor for the jumps in different directions keeping the same rates) has formally the same invariant product measures, but for that process we have no self-duality and therefore no proof of existence in infinite volume.

The rest of our paper is organised as follows. In Section 2, we give basic definitions and set notations up, in Section 3, we prove the successful coupling, in Section 4, we characterise the