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Conditioning subordinators embedded in Markov processes

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Abstract

The running infimum of a Lévy process relative to its point of issue is known to have the same range that of the negative of a certain subordinator. Conditioning a Lévy process issued from a strictly positive value to stay positive may therefore be seen as implicitly conditioning its descending ladder height subordinator to remain in a strip. Motivated by this observation, we consider the general problem of conditioning a subordinator to remain in a strip. Thereafter we consider more general contexts in which subordinators embedded in the path decompositions of Markov processes are conditioned to remain in a strip.

Keywords: Lévy processes; Conditioning; h-transform; Subordinators; Markov processes

1. Introduction

Let $\mathbb{D}$ denote the space of càdlàg functions $\omega : [0, \infty) \to \mathbb{R} \cup \{\varDelta\}$ such that, defining $\zeta = \inf\{t \geq 0 : \omega_t = \varDelta\}$, we have $\omega(t) = \varDelta$ for $t \geq \zeta$. We call $\varDelta$ the cemetery state and think of ω as killed once it enters the cemetery state. The space $\mathbb{D}$ is equipped with the Skorokhod topology and for $t \geq 0$, we write $(\mathcal{F}_t : t \geq 0)$ for the natural filtration. The process $X = (X_t : t \geq 0)$ denotes the co-ordinate process on $\mathbb{D}$ and we let $(X, \mathbb{P}_x)$ denote the law of a non-constant Lévy process started at $x \in \mathbb{R}$.

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In, what is by now considered, classical work, it was shown in [4,6] that, under mild assumptions, there exists a (super)harmonic function $h \geq 0$ such that, for $x > 0$,

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\uparrow}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \frac{h(X_t)}{h(x)} \mathbb{1}_{\{t < \tau_0^-\}}, \quad t \geq 0, \tag{1}$$

characterises the law of a Lévy process conditioned to stay non-negative, where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$. To be more precise, the resulting (sub-)Markov process, $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, also emerges through the limiting procedure,

$$\mathbb{P}_x^{\uparrow}(A) := \lim_{q \downarrow 0} \mathbb{P}_x(A, t < \mathbb{e}_q \mid \tau_0^- > \mathbb{e}_q), \quad t \geq 0, A \in \mathcal{F}_t,$$

where, for $q > 0$, $\mathbb{e}_q := q^{-1}\mathbb{e}$ such that $\mathbb{e}$ is an independent exponentially distributed random variable with unit mean. This result would normally be proved in the setting of diffusions using potential analysis. For the case of Lévy processes the analogous theory was not readily available and so the work of [4,6] is important in that it shows how excursion theory can be used instead.

In this paper, we are interested in exploring conditionings of subordinators, that is, Lévy processes with non-decreasing paths. Moreover, we are also interested in similarities that occur when conditioning subordinators that are embedded in the path decomposition of other Markov processes. In this respect, it is natural to understand how to condition a subordinator to remain below a given threshold. To see why, let us return to the setting of conditioning a Lévy process to remain non-negative and explore the effect of the conditioning on the range of the process $\underline{X}_t := \inf_{s \leq t} X_s$, $t \geq 0$.

It is well understood that there exists a local time at 0 for the process $(X_t - \underline{X}_t : t \geq 0)$, which is Markovian; see for example Chapter VI of [1]. If we write this local time process by $(L_t : t \geq 0)$ and set $L_t^{-1} = \inf\{s > 0 : L_s > t\}$, $t \geq 0$, then $H_t := X_{L_t^{-1}}$, for $L_t^{-1} < \infty$ and $H_t := -\infty$ otherwise, defines a killed stochastic process with cemetery state $\{-\infty\}$, known as the descending ladder height process, whose range $(-\infty, 0]$ agrees with that of $(\underline{X}_t : t \geq 0)$. In particular, for $x > 0$, the law of H under $\mathbb{P}_x$ is such that $S_t := x - H_t$, $t \geq 0$ is a (killed) subordinator issued from x. (In fact, the renewal function associated to this subordinator is precisely the function h in (1).) Since, for each $t > 0$, L_t^{-1} is in fact a stopping time, one may consider the conditioning associated to $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, when viewed through the stopping times $(L_t^{-1} : t \geq 0)$, to correspond to conditioning the subordinator $(S_t : t \geq 0)$, issued from x, to remain positive; or equivalently to conditioning $-H$ to remain in the interval $[0, x)$.

With this example of a conditioned subordinator in hand, we extract the problem into its natural general setting. In the next section, we show how conditioning a general subordinator to stay in a strip, say $[0, a]$ can be developed rigorously. Additionally we show that this conditioning can be seen as the combined result of choosing a point in a according to a distribution, which is built from the potential measure of the subordinator, and then further conditioning the subordinator to hit that point. Moreover, in the setting of stable subordinators, appealing additionally to the theory of self-similarity, we can interpret the conditioning as the result of an Esscher change of measure in the context of the Lamperti transform.

In the spirit of observing the relationship between conditioning a Lévy processes to stay positive and the conditioning of a key underlying subordinator in its path decomposition, we look at the case of conditioning a Markov process to avoid the origin beyond a fixed time. A key element of the associated path decomposition will be role of conditioning inverse local time at the origin to remain in the interval $[0, a)$, with $a > 0$ fixed. Finally in Section 3.1 we use the ideas from the