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Travelling wave solutions to the KPP equation with branching noise arising from initial conditions with compact support

Sandra Kliem

Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Str. 9, D-45127 Essen, Germany

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Abstract

We consider the one-dimensional KPP-equation driven by space–time white noise and extend the construction of travelling wave solutions arising from initial data $f_0(x) = 1 \wedge (-x \vee 0)$ from (Tribe, 1996) to f_0 non-negative continuous functions with compact support. As an application the existence of travelling wave solutions is used to prove that the support of any solution to the SPDE is recurrent. As a by-product, several upper measures are introduced that allow for a stochastic domination of any solution to the SPDE at a fixed point in time.

MSC: primary 60H15; secondary 35R60

Keywords: Stochastic PDE; KPP equation; White noise; Travelling wave; Initial conditions with compact support; Recurrence

1. Introduction

1.1. Motivation

Consider non-negative solutions to the one-dimensional stochastic partial differential equation (SPDE)

$$\partial_t u = \partial_{xx} u + \theta u - u^2 + u^{\frac{1}{2}} dW, \qquad t > 0, x \in \mathbb{R}, \theta > 0 \tag{1.1}$$

E-mail address: sandra.kliem@uni-due.de.

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$$u(0, x) = u_0(x) \geq 0,$$

where $W = W(t, x)$ is space–time white noise and $\theta > 0$ a parameter. The deterministic part of this SPDE is (after appropriate scaling, cf. Mueller and Tribe [13, Lemma 2.1.2]) the well-studied Kolmogorov–Petrovskii–Piskunov-(KPP)-equation (also known as the Kolmogorov- or Fisher-equation). In Bramson [2] the existence of a family of non-negative travelling wave solutions to this deterministic partial differential equation (PDE) is established. Including the noise term, one can think of $u(t, x)$ as the density of a population in time and space. Leaving out the term $\theta u - u^2$, the above SPDE is the density of a super-Brownian motion (cf. Perkins [15, Theorem III.4.2]), the latter being the high density limit of branching particle systems that undergo branching random walks. The additional term of θu models linear mass creation at rate $\theta > 0$, $-u^2$ models death due to overcrowding. In [14], Mueller and Tribe obtain solutions to (1.1) as limits of densities of scaled long range contact processes with competition. The same techniques can be extended to obtain solutions to SPDEs with more general drift-terms, see Kliem [9].

The existence and uniqueness in law of solutions to (1.1) in the space of non-negative continuous functions with slower than exponential growth $\mathcal{C}^+_{tem}$, is established in Tribe [17, Theorem 2.2]. Let $\tau \equiv \inf\{t \geq 0 : u(t, \cdot) \equiv 0\}$ be the *extinction-time* of the process. By [13, Theorem 1], there exists a critical value $\theta_c > 0$ such that for any initial condition $u_0 \in \mathcal{C}^+_c \setminus \{0\}$ with compact support and $\theta < \theta_c$, the extinction-time of u solving (1.1) is finite almost surely. For $\theta > \theta_c$, *survival*, that is $\tau = \infty$, happens with positive probability.

Let $R_0(u(t)) \equiv R_0(t) \equiv \sup\{x \in \mathbb{R} : u(t, x) > 0\}$. Then $R_0(t) = -\infty$ if and only if $\tau \leq t$. Extending arguments of Iscoe [7] one can show that $R_0(u(0)) < \infty$ implies $R_0(u(t)) < \infty$ for all $t > 0$. Using R_0 as a *(right) wavefront marker*, we look for so-called *travelling wave solutions* to (1.1), that is solutions with the properties

(i) $R_0(u(t)) \in (-\infty, \infty)$ for all $t \geq 0$, (1.2)

(ii) $u(t, \cdot + R_0(u(t)))$ is a stationary process in time. (1.3)

In [17, Section 3] the existence of travelling wave solutions to (1.1) is shown, in [17, Section 4] it is established that for $\theta > \theta_c$ any travelling wave solution has an asymptotic (possibly random) *wave speed*

$$R_0(u(t))/t \to A \in \left[0, 2\theta^{1/2}\right] \quad \text{for } t \to \infty \text{ almost surely.} \tag{1.4}$$

Strict positivity of A remains an open problem if θ is not big enough.

To construct a travelling wave, [17] proceeds as follows. Use $R_1(u(t)) \equiv \ln\left(\int \exp(x) u(t, x)dx\right)$ in place of the wavefront marker $R_0(t)$ and take as initial condition $f_0(x) \equiv 1 \wedge (-x \vee 0)$ in (1.1). Then the sequence $(\nu_T)_{T\in\mathbb{N}}$ with

$$\nu_T \quad \text{the law of } T^{-1} \int_0^T u(s, \cdot + R_1(u(s)))ds \tag{1.5}$$

is tight and any limit point ν is nontrivial. Starting in u_0 with distribution ν, shifted by R_0, one then obtains a travelling wave solution to (1.1).

The investigation of survival properties of solutions to (1.1) is a major challenge, where the main difficulty comes from the competition term $-u^2$. Without competition, the underlying "additive property" (cf. [15, pages 167–168 and 159] in the context of Dawson–Watanabe superprocesses with drift) facilitates the use of Laplace functionals. Including competition, only subadditivity in the sense of [13, Lemma 2.1.7] holds.