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# On the class of distributions of subordinated Lévy processes and bases

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**Abstract**

This article studies the class of distributions obtained by subordinating Lévy processes and Lévy bases by independent subordinators and meta-times. To do this we derive properties of a suitable mapping obtained via Lévy mixing. We show that our results can be used to solve the so-called recovery problem for general Lévy bases as well as for moving average processes which are driven by subordinated Lévy processes.


*Keywords:* Subordination; Lévy basis; Lévy processes; Lévy mixing; Lévy semistationary processes; Recovery problem

## 1. Introduction

Lévy processes and Lévy bases constitute important building blocks for constructing realistic models for temporal and/or spatial phenomena. In addition, it has often been noted that *stochastic volatility/intermittency*, which can be regarded as stochastic variability beyond the fluctuations described by the Lévy noise, is present in empirical data of e.g. asset prices in finance or turbulence in physics. One possibility of accounting for such additional variability is by using

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the concept of subordination, see [15,24] and also [33] for additional references. In this article, we are interested in characterizing the law of subordinated Lévy processes and Lévy bases. We will approach this problem from the general angle of defining a mapping

$$\Phi_M(\rho)(A) := \int_0^\infty \mu(s, A)\,\rho(ds), \quad A \in \mathcal{B}(\mathbb{R}\setminus\{0\}), \tag{1}$$

where $M = (\mu(s,\cdot))_{s\geq 0}$ denotes a measurable collection of probability measures and $\rho$ is a Lévy measure on $\mathbb{R}^+$. We note that if $\mu(s,\cdot)$ is chosen as the law of a Lévy process at time $s$ and $\rho$ is a $\sigma$-finite measure supported on the positive half line, then $\Phi_M$ describes the Lévy measure of a subordinated Lévy process (with independent subordinator). We will discuss in which sense the mapping is related to the concept of *Lévy mixing*, as introduced by Barndorff-Nielsen et al. [6].

While the mapping (1) can be defined for various measures $\mu$, we are mainly interested in the situation when $\mu(s,dx) = \mu^s(dx)$ for an infinitely divisible (ID) law $\mu$. In that case, we shorten the notation and typically write $\Phi_\mu = \Phi_M$. In particular, we focus on three scenarios in more detail: The cases when (1) $\Phi_\mu$ is restricted to the finite Lévy measures on $(0,\infty)$, (2) $\mu$ is symmetric, and (3) $\mu$ is supported on $(0,\infty)$. Our first results in this context contain a detailed description of the properties of the mapping $\Phi_\mu$. In particular, we characterize its Lévy domain and some properties of its range as well as establish conditions under which the mapping $\Phi_\mu$ is injective.

These results can then be used to describe the law of subordinated Lévy processes and bases.

As an application of our results, we study the so-called *recovery problem* for subordinated ID processes: If we observe a subordinated Lévy process $X_t = L_{T_t}$ (where $L$ is a Lévy process and $T$ is an independent subordinator), can we recover $T$ from $X$ and, if so, in which sense? In answering this question we will build upon and further extend earlier work by Winkel [35] and Geman et al. [19].

Moreover, we will go one step further and use such a subordinated Lévy process as the driving noise in a moving average type process and derive suitable conditions which allow us to recover the subordinator from observations of the moving average process. In the special case of a Brownian moving average process restricted to be a semimartingale, such a problem has been addressed by Comte and Renault [16]. More recently, Sauri [31] studied the invertibility of infinitely divisible continuous moving averages processes and we can build on this result to solve the recovery problem in the more general set-up which includes non-semimartingale processes.

While we focus on the recovery problem in the distributional sense in this article, we remark that recently there has been growing interest in related statistical problems: E.g. Belomestny [11] showed how the characteristic triplet of a multivariate Lévy process can be estimated if only low-frequency discrete observations are available from the time-changed Lévy process. Also, more recently, Belomestny and Schoenmakers [12] have studied statistical inference of a time-changed Lévy process, focusing in particular on inference for the subordinator, when discrete observations of the subordinated Lévy process are available.

The outline for the remaining article is as follows. Section 2 introduces the basic notation and background material on Lévy processes, Lévy bases, subordination and meta-times. In Section 3, we will define the mapping (1) and study its key properties. We will then use these results in Section 4 to describe the law of subordinated Lévy processes. The recovery problem for Lévy bases and moving average processes driven by subordinated Lévy processes is then studied in Section 5.