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Strikingly simple identities relating exit problems for Lévy processes under continuous and Poisson observations

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Abstract

We consider exit problems for general Lévy processes, where the first passage over a threshold is detected either immediately or at an epoch of an independent homogeneous Poisson process. It is shown that the two corresponding one-sided problems are related through a surprisingly simple identity. Moreover, we identify a simple link between two-sided exit problems with one continuous and one Poisson exit. Finally, identities for reflected processes and a link between some Parisian type exit problems are established. For spectrally one-sided Lévy processes this approach enables alternative proofs for a number of previously established identities, providing additional insight.

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1. Introduction

Let $X = (X_t,\ t \geq 0)$ be a real-valued Lévy process, and let $T_i, i \geq 1$ be the epochs of a Poisson process with intensity $\lambda > 0$ which is independent of X; also add $T_0 = 0$. The probability law corresponding to X started at u will be denoted by $\mathbb{P}_u$ (with $\mathbb{E}_u$ denoting the expectation). When u is not mentioned explicitly we assume that $u = 0$ and write simply $\mathbb{P}$ and $\mathbb{E}$. Define

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\}, \qquad \tau_a^+ = \inf\{t \geq 0 : X_t > a\},$$
$$\widehat{\tau_0^-} = \min\{T_i, i \in \mathbb{N}_0 : X_{T_i} < 0\}, \qquad \widehat{\tau_a^+} = \min\{T_i, i \in \mathbb{N}_0 : X_{T_i} > a\},$$

which we interpret as the first passage times under continuous and Poisson observations, respectively. Observe that $\tau_0^- < \widehat{\tau_0^-}$ and, moreover, $\widehat{\tau_0^-}$ converges in probability to τ_0^- as $\lambda \to \infty$ (the same is true for τ_a^+ and $\widehat{\tau_a^+}$). Thus exit theory under Poisson observation can be regarded as a generalization of the classical exit theory. Throughout this paper, however, we keep $\lambda > 0$ fixed.

Observation at Poisson epochs is both of theoretical and practical interest. Firstly, some exit problems with Poisson observation yield transforms of certain occupation times, e.g.

$$\mathbb{P}_u(\tau_0^- < \widehat{\tau_a^+}) = \mathbb{E}_u\left[\exp\left(-\lambda \int_0^{\tau_0^-} 1_{\{X_t > a\}}\mathrm{d}t\right); \tau_0^- < \infty\right], \quad u \in \mathbb{R},$$

which readily follows from the void probability formula for a Poisson process. Secondly, Poisson observation is relevant in various applications such as queueing (see e.g. [5]), reliability and insurance risk theory (see e.g. [1,2]). In particular, in many applications discrete-time observation of stochastic processes would often be considered more natural, but for equidistant discrete time epochs the explicit and tractable analytical structure of continuous-time processes is typically destroyed, so that one is forced towards numerical techniques for the determination of exit probabilities and related quantities. The Poisson observation structure is a bridge between continuous-time and discrete-time observation that still leads to rather explicit, and as will be shown below, also somewhat elegant modifications of the continuous-time formulas.

1.1. *Overview and organization*

In order to stress the intuition behind the derivation of the identities, we will start with a simple case and gradually generalize the setup. Most of the results are stated in terms of relations between transforms, but can also be understood as relations between the corresponding laws in an obvious way.

Some of the wording throughout the manuscript will be in terms of the insurance application, where X is the surplus process of a portfolio of insurance contracts, τ_0^- is the time of ruin of the portfolio, $\{\tau_0^- = \infty\}$ is the event of (infinite-time) survival, and $\widehat{\tau_0^-}$ is the time of observed ruin under Poisson observation of the surplus process (in the application the Poisson epochs can for instance be interpreted as the observation times of the regulatory authority).

In Section 2 we discuss survival probabilities corresponding to the two observation types, and then proceed to the general one-sided exit problems including the time of exit and the overshoot. In Section 3 we consider more complex problems. Firstly, the two-sided exit problem with one continuously observed and one (Poisson-)discretely observed boundary is related to the one where the observation types at the boundaries are interchanged. Secondly, we provide a link between Poisson exit of a reflected process and continuous exit of the process reflected at Poisson