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The quasi-linear method of fundamental solution applied to transient non-linear Poisson problems

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A R T I C L E I N F O

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A B S T R A C T

This paper proposes the use of a quasi-linear method of fundamental solution(QMFS) and explicit Euler method to treat the transient non-linear Poisson-type equations. The MFS, which is a fully meshless method, often deals with the linear and non-linear poisson equations by approximating a particular solution via employing radial basis functions (RBFs). The interpolation in terms of RBFs often leads to a badly conditioned problem which demands special cares. The current work suggests a linearization scheme for the nonhomogeneous term in terms of the dependent variable and finite differencing in time resulting in Helmholtz-type equations whose fundamental solutions are available. Consequently, the particular solution is no longer needed and the MFS can be directly applied to the new linearized equation. The numerical examples illustrate the effectiveness of the presented method.

1. Introduction

Transient non-linear Poisson problems are widely encountered in the modeling of physical phenomena. For example, transient heat conduction or mass diffusion with source terms arises in model equations in many different areas of computational physics and engineering. Representative prototype problems include transient diffusion with chemical reaction in a catalyst pellet, microwave heating process, spontaneous combustion, and thermal explosion problems and transient convection.

The numerical solution procedure usually depends on finite-difference, finite-element, boundary-element or spectral methods. The boundary element method (BEM) is one such method suitable for linear problems [1,2]. For transient problems, the BEM can be used in conjunction with finite differencing in time [3,4]. The resulting formulation is a steady-state type of Poisson equation that can be solved by dual reciprocity methods (DRM) [1]. Thus, the advantages of the boundary only discretization are retained, and the internal points are needed only for the interpolation of the nonhomogeneous terms. The disadvantage of all the BEM-based techniques is the need for the evaluation of singular or near-singular integral which can be time consuming.

As an alternative, solution methods based on the method of fundamental solution (MFS) are gaining considerable attention [5,6]. These methods are based on fitting of the boundary conditions with the fundamental solutions of the Laplace equation as the basis functions [7,8]. The poles or singularities of the fundamental solutions are placed outside the domain, thus avoiding the need for evaluation of the singular integrals in contrast to traditional BEM. Similar to the BEM the MFS is in disadvantage when the fundamental solution of the underlying equation is not available. In this case a part of the equation, whose fundamental solution is provided, is considered as a homogeneous equation for which the MFS can be directly applied and the global solution is then obtained by assembling a particular solution and the homogeneous solution [9,10]. This method has been demonstrated for various linear differential equations and in conjunction with the method of particular solutions for non-linear Poisson problems [11,12]. In view of the rapid development of the MFS-RBF method in recent years, the applications to transient problems would be interesting. However, the application of the MFS-RBF method to transient problems has been limited. For linear transient problems, procedures based on finite differencing in time need to be used [13]. The simplest method is to use an explicit Euler method for approximating the time derivatives, and a paper by Golberg and Chen [14] provides a detailed computational study based on this approach. Also, the forcing function f was approximated at the previous time step in their study. The explicit scheme presented in their study is first-order accurate and has stability restrictions.

The use of RBFs usually leads to an ill-conditioned problem. There have been some approaches including preconditioning, locally supported RBFs [15] and domain decomposition [16] to treat the conditioning. In addition, the use of Hermite interpolation, called osculatory RBF (OS-RBF) has been proposed to improve the interpolation quality [13,17].

To avoid ill-conditioning, this work proposes a quasi-linear MFS (QMFS) and explicit Euler method for approximating the time derivatives for non-linear transient Poisson problems, as a

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function of the dependent variable, is linearized [18,19]. The resulting linear terms together with the homogeneous term form a Helmholtz equation whose fundamental solution is already known. As a consequence, the particular solution method is avoided and the use of RBFs is not required.

A brief outline of this paper is as follows. In Section 2, the MFS for linear PDEs is reviewed. A generalization of the QMFS for non-linear Poisson equations is discussed in Section 3. The Euler method will be described and applied to QMFS to deals with heat equations in Section 4. Some numerical experiments will be presented in Section 5.

2. The method of fundamental solution

In this section we briefly describe the MFS for the homogeneous PDE [7]

$$Lu(P) = 0, \quad P \in \Omega, \tag{1}$$

with the combined BCs

$$\text{Dirichlet}: u = \overline{u} \quad \text{over } \Gamma_1,$$

$$\text{Neumann}: \frac{\partial u}{\partial n} = \overline{q} \quad \text{over } \Gamma_2, \tag{2}$$

where L is a linear differential operator, Ω is a bounded domain in $\Re^2$ or $\Re^3$, enclosed by Γ, $\Gamma_1+\Gamma_2=\Gamma$ and $\overline{u}$ and $\overline{q}$ are known functions. The major tool in the MFS is the fundamental solution used in the classical BEM. The fundamental solution of Eq. (1) is a function $G(P,Q)$ which satisfies

$$LG(P,Q) = -\delta(P,Q), \quad P,Q \in \Re^n,\ n = 2,3, \tag{3}$$

where $\delta(P,Q)$ represents the dirac delta function acting at point Q. For instance when L is the Laplace operator, $G(P,Q)$ is given by $G(P,Q)=(1/2\pi)\ln 1/\|P-Q\|$ and $G(P,Q)=1/4\pi\|P-Q\|$, respectively, for two- and three-dimensional cases. The function $G(P,Q)$ is equal to zero everywhere except when $P=Q$ where it is singular and goes to infinity. The main idea in the MFS is to express the solution in terms of the fundamental solutions as

$$u(P) = \sum_{i=1}^{n} a_i G(P,Q_i), \tag{4}$$

or in a simpler form

$$u(P) = \sum_{i=1}^{n} a_i G(r_i), \tag{5}$$

where $r_i = \|P-Q_i\|$, Q_i's represent the source points and P is any point under consideration. Since $G(r_i)$ satisfies (3) for any source point Q_i for which $r_i \neq 0$, $LG(r_i)=0$ is satisfied. Consequently, the function $u(P)$ in (5) exactly satisfies the PDE in (1) for any a_i provided that the source points are selected in a way that $r_i \neq 0$. In order to achieve this goal, a virtual boundary is employed and the source points Q_i are selected on this boundary (see Fig. 1).

It should be mentioned that effectiveness of MFS depends strongly on shape of virtual boundary and distance between the virtual and physical boundary [6]. Any shape of the virtual boundaries can be, theoretically, used in the calculation. However, due to the limitation of computers inherent precision, the shape of the virtual boundary may influence the numerical accuracy of the output results. It is proved that circular virtual boundary and similar virtual boundary are suitable for MFS. Based on these two schemes, for example, the shapes of a virtual boundary can be chosen as either rectangle or circle for a rectangular domain. From the point of view of computation, accuracy of the numerical results will become worse if the distance between the virtual boundary and physical boundary is too close, that may cause problems due to singularity of the fundamental solutions. Conversely, round-off error in C/Fortran floating point arithmetic may be a serious problem when the source points are far from the real boundary.

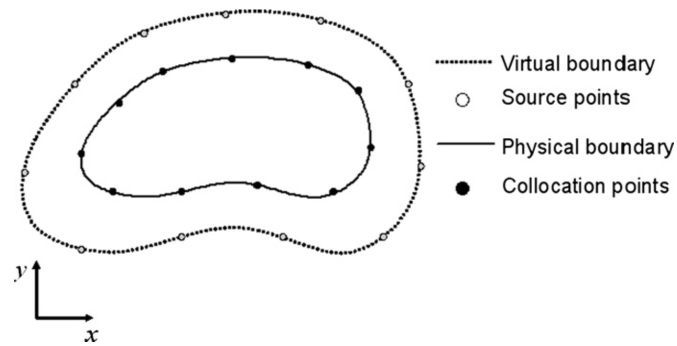

Fig. 1. The physical and virtual boundary with the collocation and source points displayed.

Since the function $u(P)$ is only an approximation to the solution on the boundary, the boundary residual can be introduced as below:

$$R_1: \ u-\overline{u} \neq 0 \quad \text{over } \Gamma_1,$$

$$R_2: \ \frac{\partial u}{\partial n}-\overline{q} \neq 0 \quad \text{over } \Gamma_2. \tag{6}$$

To evaluate the unknown parameters a_i, there are some approaches, two important of which are collocation [6] and Galerkin methods [20]. In the collocation method, which is employed in this work, n collocation nodes are selected on the real boundary (see Fig. 1) and the residual is set to zero at each collocation point. This results in a linear system of equations whose solution provides the unknown values of a_i. In the Galerkin method, the residual is forced to zero by a weighted residual technique using G_i as weighting functions, that is

$$\int_{\Gamma_1} G_i(u-\overline{u})\,d\Gamma + \int_{\Gamma_2} G_i\left(\frac{\partial u}{\partial n}-\overline{q}\right)d\Gamma = 0,$$

which again leads to a linear system of equations.

3. Quasi-linear MFS

We now briefly describe the QMFS and apply a linearization technique to deal with the non-linear Poisson-type equations [19]. We consider the Poisson equation

$$\nabla^2(u) = f(u), \quad \text{with the boundary conditions} \begin{cases} u=\overline{u} & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial n}=\overline{q} & \text{on } \Gamma_2, \end{cases} \tag{7}$$

where $\Gamma = \Gamma_1+\Gamma_2$ is the boundary of the problem. Using a linear Taylor polynomial, we quasi-linearize the non-linear function $f(u)$ over the domain Ω at $\tilde{u}$, that is

$$f(u) = f(\tilde{u}) + (u-\tilde{u})\left.\frac{df}{du}\right|_{\tilde{u}}, \quad \text{or} \quad f(u) = k_1 + k_2 u, \tag{8}$$

where $\tilde{u}$ is a suitably defined average value of u over Ω, and k_1 and k_2 are constants quasi-linearization which are specified as follows:

$$k_1 = f(\tilde{u}) - \left.\left(\frac{df}{du}\right)\right|_{\tilde{u}} \tilde{u} \quad \text{and} \quad k_2 = \left.\left(\frac{df}{du}\right)\right|_{\tilde{u}}. \tag{9}$$

Using (8), Eq. (7) is converted to

$$\nabla^2(u) = k_1 + k_2 u. \tag{10}$$