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Regular hybrid boundary node method for biharmonic problems

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ABSTRACT

The regular hybrid boundary node method (RHBNM) is a new technique for the numerical solutions of the boundary value problems. By coupling the moving least squares (MLS) approximation with a modified functional, the RHBNM retains the meshless attribute and the reduced dimensionality advantage. Besides, since the source points of the fundamental solutions are located outside the domain, 'boundary layer effect' is also avoided. However, an initial restriction of the present method is that it is only suitable for the problems which the governing differential equation is in second order.

Now, a new variational formulation for the RHBNM is presented further to solve the biharmonic problems, in which the governing differential equation is in fourth order. The modified variational functional is applied to form the discrete equations of the RHBNM. The MLS is employed to approximate the boundary variables, while the domain variables are interpolated by a linear combination of fundamental solutions of both the biharmonic equation and Laplace's equation. Numerical examples for some biharmonic problems show that the high accuracy with a small node number is achievable. Furthermore, the computation parameters have been studied. They can be chosen in a wide range and have little influence on the results. It is shown that the present method is effective and can be widely applied in practical engineering.

1. Introduction

Many physical problems are modeled by the biharmonic equation, particularly those arising in fluid dynamics and elasticity problems. For example, the governing equations of Stokes flow problems and flows through porous media are biharmonic functions. The biharmonic functions also arise when dealing with the transverse displacements of plates and shells. As the geometrical, boundary and loading conditions may be very complex, it is usually difficult to obtain the analytical solutions. Therefore, the studies on numerical methods for solving this kind of biharmonic equation make great significance to practical engineering.

In many cases, the boundary element method (BEM) has been applied for solving the biharmonic problems more than the methods of domain types, e.g. the finite element method (FEM) or the finite difference method (FDM). The indirect BEM was first used by Jaswon et al. [1]. Later, Kelmanson [2] applied a direct BEM to a variety of biharmonic problems which involve boundary singularities. However, the BEM still requires boundary discretization, which may cause some inconvenience in the implementation, such as tackling complicated boundary problems and moving boundary problems.

In recent years, some novel computational methods called meshless methods have been developed. These methods do not require elements and thus attract more and more attention. They have many advantages, such as flexibility, efficiency and versatility for complex geometry. The meshless methods are a great variety and may be divided into two categories: domain-type and boundary-type methods. Several domain-type meshless methods, such as: the diffuse element method (DEM) [3], the element-free Galerkin (EFG) method [4], the reproducing kernel particle method (RKPM) [5], the point interpolation method (PIM) [6], the meshless local Petrov–Galerkin (MLPG) method [7,8], have been proposed and achieved remarkable progress in solving a wide range of practical engineering problems. The boundary-type meshless methods proposed include the local boundary integral equation (LBIE) method [9], the boundary node method (BNM) [10], the boundary point interpolation method (BPIM) [11], the boundary element-free method (BEFM) [12], the Galerkin boundary node method (GBNM) [13] and the boundary face method (BFM) [14].

The aforementioned meshless methods do not need an element mesh for the interpolation of the field or boundary variables, but some of them have to use background cells for integration. The requirement of background cells for integration makes the method being not "truly" meshless.

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Zhang et al. [15] proposed another boundary-type meshless method: the hybrid boundary node method (HBNM). It gets rid of the background elements and achieves a truly meshless method. It uses the MLS to approximate the boundary variables, and the integration is limited to a fixed local region. Elements are required neither for interpolation nor integration. The HBNM has been used to solve the potential problems [16] and elasticity problems [17]. However, it has a drawback of serious 'boundary layer effect', i.e., the accuracy of the results near the boundary is very sensitive to the proximity of the interior points nearby the boundary. To avoid this shortcoming, Zhang et al. [18,19] further proposed the regular hybrid boundary node method (RHBNM), in which the source points of the fundamental solutions are located outside the domain rather than on the boundary.

So far, these two methods can only be used for solving certain elliptic boundary value problems, which the governing differential equation is in second order, and have never been applied to solve the fourth order biharmonic problems.

In this paper, a new variation formulation for the RHBNM is presented to solve biharmonic problems. The RHBNM is based on a hybrid displacement variational principle and the MLS approximation. For the biharmonic problems, four independent variables: field function in the domain u, field functions on the boundary $\tilde{u}$, $\tilde{q}=\tilde{u}_{,n}$ and $\tilde{M}=\nabla^2\tilde{u}$ are used in the modified functional. Then, a new hybrid displacement variation formulation is developed. The present method interpolates the domain variables using a linear combination of fundamental solutions of both the biharmonic equation and Laplace's equation. The unknown boundary variables, same as the RHBNM for second order elliptic boundary value problems, are approximated by the MLS method. Therefore, the RHBNM for biharmonic problems is achieved.

This paper is organized as follows: the RHBNM for biharmonic problems is formulated in Section 2. Numerical examples for the 2-D biharmonic problems are given in Section 3. Finally, the paper ends with conclusions in Section 4.

2. Development of the RHBNM for biharmonic problems

In this section, the following biharmonic problem is considered:

$$\nabla^2(\nabla^2 u)=\nabla^4 u=0 \quad \text{in} \quad \Omega \tag{1}$$

$$u=\overline{u}, \frac{\partial u}{\partial n}\equiv q=\overline{q} \quad \text{on} \quad \Gamma_1 \tag{2a}$$

$$u=\overline{u}, \nabla^2 u\equiv M=\overline{M} \quad \text{on} \quad \Gamma_2 \tag{2b}$$

where the domain Ω is enclosed by $\Gamma=\Gamma_1+\Gamma_2$; $\overline{u}$, $\overline{q}$ and $\overline{M}$ are the prescribed functions and n the unit outward normal.

2.1. Variational principle

The total potential energy can be given as

$$\Pi_p=\int_\Omega \frac{1}{2}u_{,ii}u_{,jj}\,\mathrm{d}\Omega-\int_{\Gamma_2}\overline{M}q\,\mathrm{d}\Gamma \tag{3}$$

As pointed out as before, there are four independent variables in the variational principle: field function in the domain u, field functions on the boundary $\tilde{u}$, $\tilde{q}=\tilde{u}_{,n}$ and $\tilde{M}=\nabla^2\tilde{u}$. Eq. (3) should also satisfy the boundary compatibility conditions

$$u=\tilde{u} \quad \text{on} \quad \Gamma \tag{4a}$$

$$q=\tilde{q} \quad \text{on} \quad \Gamma \tag{4b}$$

where u and q are the field functions in the domain but very close to the boundary.

Introducing the compatibility conditions of Eq. (4) into the functional expression of Eq. (3), the modified variational functional can be obtained

$$\Pi_p^*=\int_\Omega \frac{1}{2}u_{,ii}u_{,jj}\,\mathrm{d}\Omega-\int_{\Gamma_2}\overline{M}\tilde{q}\,\mathrm{d}\Gamma+\int_\Gamma (u-\tilde{u})\frac{\partial\tilde{M}}{\partial n}\,\mathrm{d}\Gamma-\int_\Gamma (q-\tilde{q})\tilde{M}\,\mathrm{d}\Gamma \tag{5}$$

Taking the variational of Eq. (5), we have

$$\delta\Pi_p^*=\int_\Omega u_{,iijj}\delta u\,\mathrm{d}\Omega+\int_\Gamma (M-\tilde{M})\delta\tilde{q}\,\mathrm{d}\Gamma-\int_\Gamma\left(\frac{\partial M}{\partial n}-\frac{\partial\tilde{M}}{\partial n}\right)\delta u\,\mathrm{d}\Gamma$$
$$-\int_\Gamma (q-\tilde{q})\delta\tilde{M}\,\mathrm{d}\Gamma+\int_\Gamma (u-\tilde{u})\delta\frac{\partial\tilde{M}}{\partial n}\,\mathrm{d}\Gamma+\int_{\Gamma_2}(\tilde{M}-\overline{M})\delta\tilde{q}\,\mathrm{d}\Gamma \tag{6}$$

Let $\delta\Pi_p^*=0$, the following integration equations can be obtained as

$$\int_\Omega u_{,iijj}\delta u\,\mathrm{d}\Omega-\int_\Gamma\left(\frac{\partial M}{\partial n}-\frac{\partial\tilde{M}}{\partial n}\right)\delta u\,\mathrm{d}\Gamma=0 \tag{7}$$

$$\int_\Gamma (u-\tilde{u})\delta\frac{\partial\tilde{M}}{\partial n}\,\mathrm{d}\Gamma=0 \tag{8}$$

$$\int_\Gamma (q-\tilde{q})\delta\tilde{M}\,\mathrm{d}\Gamma=0 \tag{9}$$

$$\int_\Gamma (M-\tilde{M})\delta\tilde{q}\,\mathrm{d}\Gamma=0 \tag{10}$$

$$\int_{\Gamma_2}(\tilde{M}-\overline{M})\delta\tilde{q}\,\mathrm{d}\Gamma=0 \tag{11}$$

Because $\overline{u}$ is the prescribed function on the whole boundary $\Gamma=\Gamma_1+\Gamma_2$, $\delta\tilde{u}=0$ and the second integral are vanished in Eq. (7). Then Eq. (1) is substituted, Eq. (7) can be satisfied. If the boundary condition $\tilde{M}=\overline{M}$is imposed, Eq. (11) will also be satisfied. So Eqs. (7) and (11) can be ignored temporarily in the following development.

It can be seen that Eqs. (8)–(10) hold for any portion of the domain Ω, for example, in a sub-domain Ω_s, which is bounded by Γ_s and L_s (Fig. 1). Following Refs. [7,9], the weak forms on a sub-domain Ω_s and its boundaries Γ_s and L_s are used to replace Eqs. (8)–(10). The test function $v_J(Q)$ is used to replace the variational part. They can be presented as

$$\int_{\Gamma_s+L_s}(u-\tilde{u})v_J(Q)\mathrm{d}\Gamma=0 \tag{12}$$

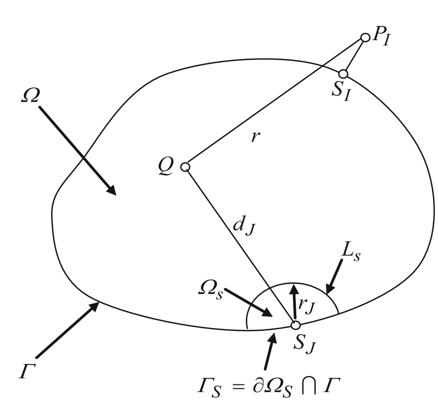

Fig. 1. Local domain centered at node s_J and source point of fundamental solution corresponding to node s_I.