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Theoretical equivalence in classical mechanics and its relationship to duality

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ABSTRACT

As a prolegomenon to understanding the sense in which dualities are theoretical equivalences, we investigate the intuitive 'equivalence' of hyper-regular Lagrangian and Hamiltonian classical mechanics. We show that the symplectification of these theories (via Tulczyjew's Triple) provides a sense in which they are (1) isomorphic, and (2) mutually and canonically definable through an analog of 'common definitional extension'.

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1. Introduction

In this paper, we will show that a careful consideration of the relationship between 'physical dualities' and the more general philosophical theme of 'theoretical equivalence' leads to a deeper understanding of both topics. We do so by focusing on the equivalence between the *hyper-regular* case of classical Lagrangian and Hamiltonian mechanics: call this **CM-Equiv**. On the one hand, this example offers us insight into what it might mean for two dual theories to be *theoretically equivalent*, in a sense that goes beyond the (admittedly impressive) 'dictionary correspondences' which are characteristic of a physical duality. On the other hand, our work will bring the (in many cases highly abstract) philosophical discussion about theoretical equivalence to bear on a concrete physical example, viz. CM-Equiv.

The rest of this introduction focuses on the theme of 'theoretical equivalence'. Section 1.1 recalls the fact that exact dualities are theoretical equivalences, and Section 1.2 reviews the discussion of theoretical equivalence in the philosophy of science literature. Section 1.3 then lays out our strategy for showing that the correspondence between (hyper-regular) Hamiltonian and Lagrangian mechanics is a theoretical equivalence.

In Section 2, we briefly review the mathematical background to our discussion. Section 2.1 recalls the 'standard' framework of classical mechanics and considers problems with describing theoretical equivalence within this formalism. Section 2.2 then sketches Tulczyjew's (cf. Tulczyjew, 1977 and related work) reformulation of classical mechanics, which we take to be an improvement on the standard framework.

In Section 3, we argue that Tulczyjew's reformulation can be interpreted as establishing a theoretical equivalence between Hamiltonian and Lagrangian mechanics (CM-Equiv). Section 3.1 reviews the notion of Common Definitional Extension (CDE) which is often deployed in the philosophy of science literature as a criterion of theoretical equivalence. In Sections 3.2 and 3.3, we go on to discuss two ways in which Tulczyjew's results can be interpreted as saying that Lagrangian and Hamiltonian mechanics have a CDE (these correspond to the two different notions of 'theory' – T1 and T2 – discussed in Section 2.1 below). Section 3.4 then

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discusses the relationship between (CM-Equiv) and various themes from category theory. In particular, we discuss how the Tulczyjew triple is natural in a category-theoretic sense and explain how this points to an analogy with the notion of Morita equivalence in ring theory.

Finally, Section 4 lists the rich connections and analogies between our interpretation of CM-Equiv and more sophisticated notions of duality in physics.

1.1. Duality as theoretical equivalence

As discussed by the other articles in this issue, modern physics – especially string theory and QFT – has uncovered a spectacular array of dualities.[1] These take the form of an (in many cases only conjectured) 'equivalence' between the dual theories, which manifests itself as a bijection (also called a 'dictionary correspondence' or 'duality map') between the physical quantities (and states) of each theory.

This bijection between physical quantites has justly attracted philosophical attention because of its remarkable features. For instance, it often exchanges the weak and strong coupling regimes of a pair of seemingly different theories, thus raising interpretive questions about how we should understand 'fundamentality' in such theories. On the other hand, it is clear that the bijection between physical quantities does not by itself constitute a theoretical equivalence, since it does not tell us how to map the *entire* (mathematical) structure of one theory to another. One can thus pose a fundamental philosophical question that goes beyond the dictionary aspect of dualities:

> (Dual-Equiv) What does it mean for a pair of dual theories to be equivalent?

In what follows, we will approach Dual-Equiv by focusing on a special case, viz. the equivalence between (hyper-regular) Lagrangian and Hamiltonian mechanics, which we have called CM-Equiv above. Of course, CM-Equiv is not usually characterized as an instance of physical duality, due to its being too elementary. Nonetheless, the problem of adequately characterizing the equivalence between these simple theories is less simple than what one might imagine, and its resolution brings together interesting tools from symplectic geometry and category theory. It may also provide a template for how to approach theoretical equivalence in more sophisticated cases.

Furthermore, our analysis of CM-Equiv illustrates how one might begin to connect the topic of duality to more general discussions about 'theoretical equivalence' in the philosophy of science, to which we turn in Section 1.2.

1.2. Theoretical equivalence in the philosophy of science

The question of theoretical equivalence has two parts: (A) What is a scientific theory?; which in turn allows us to pose: (B) When are two scientific theories *equivalent*? A long and often tortuous literature has grown out of all the attempts to give satisfactory answers to (A) and (B).

Beginning with the logical positivists, through Carnap and Hempel, the so-called *syntactic* view of theories was developed.[2] Roughly, this view answers (A) by claiming that a theory is a collection of theorems expressed in a fixed formal language, and it answers (B) by claiming that two theories are equivalent iff they are logically equivalent, in the sense that each theory's axioms can be derived from the other's. There were several problems with this view. First, undue emphasis was placed on attempting to spell out the scientific application of properties of formal systems (e.g. axiomatizability, decidability, and interpolation). Second, and more importantly for our purposes, identifying two scientific theories only if they are logically equivalent proves far too restrictive: in particular, this means that one cannot even compare two theories that are formulated using different vocabularies (i.e. have different *signatures* in the model-theoretic sense.)

In reaction to the problems with the syntactic view, the *semantic* view of theories was developed by, among others, van Fraassen, Suppe and Glymour (cf. e.g. van Frassen, 1980; Suppe, 1989). Roughly speaking, this view holds, in answer to (A), that a theory is a class of models, and the theory is true iff one of these models is 'isomorphic' to the actual world. Thus, in answer to (B), the semantic view holds that two such theories will be considered equivalent iff they have identical classes of models. There are also several problems with this view,[3] but the one that concerns us the most is the following: What is a good notion of 'equivalence' between two classes of models? In other words, what kind of relation must these classes stand in if we are to consider them equivalent?[4]

This all-too-brief sketch[5] is sufficient to show that efforts to answer the question of theoretical equivalence have resulted in a dialectical limbo. For both the last century's major approaches to answering this question seem to face insurmountable difficulties. But our aim here is neither to find some dialectical middle ground nor to ameliorate any of the currently available options. Instead, we are motivated by a more practical challenge: the philosophy of science has still not been able to provide a fruitful and appealing framework for conceptualizing the simplest cases of 'intuitive theoretical equivalence' in physics. So if a link is to be drawn between the question of theoretical equivalence in the philosophy of science, on the one hand, and ongoing research on dualities in physics, on the other, then one should first develop the resources to address simple cases such as classical mechanics.

1.3. Strategy and thesis

In order to address this practical challenge, this paper will describe a concrete and satisfactory framework for CM-Equiv. This will serve as a prolegomenon to developing a general framework for theoretical equivalence in the philosophy of science, and thus to the study of more sophisticated dualities.

By 'satisfactory', we have in mind two desiderata:

- S1: Our framework should address the question of how to construct an appropriate mathematical notion of equivalence (e.g. 'isomorphism' or any of its (n-)categorical generalizations) between dual (or 'intuitively equivalent') physical theories.
- S2: Our framework should satisfy certain independently-motivated criteria from the philosophy of science about what it means for two theories to be 'equivalent'.

Even within the elementary setting of classical mechanics, the task of exhibiting a framework which satisfies S1 and S2 is decidedly non-trivial. Let us briefly sketch how these desiderata arise in our work.

First, S1. One might imagine classical mechanics to be perfectly well-understood. But philosophers have managed to make trouble even in the mechanical paradise of our 19th century precursors:

[1] See Polchinski (2015) in this volume for an overview.

[2] Also once called the 'received view', cf. e.g. Suppe (1989).

[3] In particular, what does it mean for a model to be isomorphic to the world? What kind of epistemic import do non-world-isomorphic models have? And so on.

[4] For a thorough examination of these issues as well as a proposed strategy to tackle them cf. Halvorson (2012).

[5] For a more thorough overview of the debate, cf. Suppe (1998).