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OV or TOV?

İbrahim Semiz

Boğaziçi University, Department of Physics Bebek, İstanbul, Turkey

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ABSTRACT

The well-known equation for hydrostatic equilibrium in a static spherically symmetric spacetime supported by an isotropic perfect fluid is referred to as the Oppenheimer–Volkoff (OV) equation or the Tolman–Oppenheimer–Volkoff (TOV) equation in various General Relativity textbooks or research papers. We scrutinize the relevant original publications to argue that the former is the more appropriate terminology.

1. Introduction

The concept of a perfect fluid, one with no viscosity or heat conduction, is an idealization that serves as a simplifying assumption in many problems, both in pre-relativistic and relativistic physics. Despite being an approximation, it can lead to quite realistic results in appropriate contexts. At the down-to-Earth scale, the ideal gases of thermodynamics qualify as perfect fluids, and at the very large scale, the universe is taken to be filled with a cosmic perfect fluid whose “atoms” are the galaxies. At the intermediate scale, the plasma that makes up stars also behaves as a perfect fluid to a good approximation, and the same is assumed for stellar bodies other than regular stars, i.e. white dwarfs and neutron stars.

While small celestial objects (asteroids or small satellites) can have irregular shapes, large ones are round. The reason is quite intuitive: deviation from spherical shape will include regions higher than the average radius, that is, mountains; but a mountain will tend to spread due to its own weight. On small solid (rocky and/or icy) objects, gravity is weak, so the strength of the materials can withstand it, but on objects larger than a few thousand km in diameter, gravity working over millions of years will smooth any mountain or depression.

A celestial object made of a perfect fluid, a “star”, by definition will not be able to support shear stresses, therefore mountains; hence is expected to be spherical if static, a very intuitive argument for sphericity. Yet it has proven surprisingly difficult to rigorously prove spherical symmetry of static perfect fluid stars. For Newtonian gravity, the proof was given in Carleman (1919); Lichtenstein (1919) for General Relativity (GR), a complete proof still does not exist. The conjecture was first explicitly stated in Lichnerowicz (1955), proven for a certain class of equations-of-state (EoS – pressure–density relationships) in Masood-ul-Alam (2007), and for a wider class in Pfister (2011).

When one makes the assumption of spherical symmetry in GR, the ansatz for the spacetime metric simplifies greatly, hence, so do Einstein's equations. The problem of finding exact solutions for the spacetime metric of such a perfect-fluid body was first addressed in 1939, in back-to-back papers published in *Physical Review* by Tolman (1939) and Oppenheimer and Volkoff (1939), hence the relevant equation, to be rederived in Section 4, is called the Oppenheimer–Volkoff (OV) or Tolman–Oppenheimer–Volkoff (TOV) equation. The question of the appropriate choice is the subject of the present paper.

Of the two dominant textbooks of General Relativity published in early seventies, “MTW” (Misner, Thorne, & Wheeler, 1973) calls the equation [Eq. (23.22) of that book] OV, the other, Weinberg (1972), displays the equation [Eq. (11.1.13) of that book] without giving it a name, or providing a reference [It *does* use the term “Oppenheimer–Volkoff limit” in a different, but related context, though]. The dominant GR textbook of the next decade, Wald (1984), calls it [Eq. (6.2.19) of that book] the TOV equation, as does the quite recent tome of Zee (2013) [Eq. (13) in Section VII.4 of that book].

In the research literature, the first use of the equation we could find after 1939 is in Cameron (1959), without giving the equation a name, but crediting (Oppenheimer & Volkoff, 1939); the use in

E-mail address: ibrahim.semiz@boun.edu.tr

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Ambartsumyan and Saakyan (1961) and Bell (1961) [Eq. (2.3) and chap. 5, Eq. (5) of the respective works] is similar. The work Misner & Sharp (1964) is the first to use the phrase "Oppenheimer–Volkoff equations of hydrostatic equilibrium" but does not display the equation(s). The review article (Fowler, 1964) in the same year displays the equation, without assigning a name to it, and only very indirectly crediting Oppenheimer and Volkoff (1939) (but not Tolman); and a report (Colgate & White, 1964) credits them explicitly. Starting in 1965, such references increase in frequency; and in 1968, the first papers appear that both display the equation, and refer to it by name; except that in two papers, Tolman's name is also associated with the equation: the work (Sedrakyan & Chubaryan, 1968) calls a set of displayed equations the "Oppenheimer–Volkoff equation", whereas (Hartle & Thorne, 1968) calls it the "Tolman–Oppenheimer–Volkoff equation of hydrostatic equilibrium" and Bludman and Ruderman (1968), the "Tolman–Oppenheimer–Volkoff condition". In the same year, Wheeler and Hansen (1968) use the phrase "Tolman–Oppenheimer–Volkoff general relativistic equation of hydrostatic equilibrium" without displaying the equation.

In 1969, we could find no use of the phrase "OV equation" or equivalent, and two uses of "TOV equation" or equivalent. The corresponding numbers are one and one for 1970, two and two for 1971, none and four for 1972, and none and four for 1973. We end the year-by-year breakdown with 1973, the year of publication of the influential book "MTW", and give a decade-by-decade breakdown in Table 1.[1] We also note that one of the authors of that appropriately gravitating tome used "OV" in a previous publication (Misner & Sharp, 1964) and two of them "TOV" in a publication and a talk (Hartle & Thorne, 1968), (Ruffini & Wheeler, 1970), yet in the book they used "OV". On the other hand, the table tells us that the use of TOV has always been more popular, but started to really dominate in the last two decades.

To discuss the question of more appropriate usage, in the next section we describe the setting of the problem. In Section 3, we review and discuss the first of the relevant papers by Tolman (1939) , and in Section 4, we review and discuss the second one by Oppenheimer and Volkoff (1939). In the final section, we evaluate and conclude.

2. The setting

The question at hand is finding valid solutions for the contents and structure of a static spherically symmetric spacetime filled with an isotropic perfect fluid. Of course, the spacetime structure is assumed to be described by a metric, which obeys Einstein's Field Equations (EFE)

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ the stress–energy–momentum (SEM) tensor, and κ the coupling constant, including Newton's constant G. The relation between $G_{\mu\nu}$ and the metric, $g_{\mu\nu}$, is given in any GR textbook, e.g. Misner et al. (1973), and we use that book's sign conventions for the definitions of the intermediate mathematical objects, that is, the Riemann and Ricci tensors, and the Christoffel symbols.

[1] The searches were performed using Google Scholar (GS). For the first row, the phrase "Oppenheimer–Volkoff" was searched for, the expressions "Oppenheimer–Volkoff limit", and of course "Tolman–Oppenheimer–Volkoff" were explicitly vetoed, and the appropriate uses of the expression were counted by visually inspecting the GS blurbs; the numbers for the last two decades were estimated as roughly 75% of the total count, based on experience with previous decades. For the second row, the exact expression was searched, which left out some misspellings (or misidentifications of GS's OCR software) of Tolman, and a few instances of "Oppenheimer–Volkoff–Tolman" that we had encountered during the scrutiny for the first row; and added very few extras like "Tolman–Oppenheimer–Volkoff solution"; therefore should be approximately correct.

Table 1
Approximate numbers of occurrence of the expressions "Oppenheimer–Volkoff equation" (and equivalents) and "Tolman–Oppenheimer–Volkoff" in the research literature in each decade since the 1960's.

Name	Decade					
	60's	70's	80's	90's	00's	10's
OV	2	11	51	102	~200	~140
TOV	6	48	52	181	830	1420

The most general form for the line element of a static spherically symmetric spacetime is

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric of a two-sphere; and a perfect fluid is characterized by a stress–energy–momentum tensor of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{3}$$

where ρ and p are the energy density and the pressure, respectively, as measured by an observer moving with the fluid, and u_μ is its four-velocity; in units such that the speed of light $c=1$. Since the fluid is at rest, we have

$$u^\mu = u^0 \delta_0^\mu \tag{4}$$

and of course, the four-velocity is normalized such that

$$u_\mu u^\mu = -1 \tag{5}$$

After these preliminaries, the Einstein equations can be written down. The nontrivial components (00, 11 and 22) are

$$\frac{B}{r^2}\left(1 - \frac{1}{A} + \frac{rA'}{A^2}\right) = \kappa B \rho(r) \tag{6}$$

$$\frac{1}{r^2}\left(1 - A + \frac{rB'}{B}\right) = \kappa A p(r) \tag{7}$$

$$\frac{r}{2A}\left[-\frac{A'}{A} + \frac{B'}{B} - \frac{rA'B'}{2AB} - \frac{rB'^2}{2B^2} + \frac{rB''}{B}\right] = \kappa r^2 p(r) \tag{8}$$

where $A(r)$ and $B(r)$ are written as A and B for brevity, and prime denotes r-derivative (the 33 component is simply the 22 component multiplied on both sides by $\sin^2(\theta)$).

Alternatively, these equations can be combined to give

$$\frac{p'}{\rho + p} + 2\frac{B'}{B} = 0 \tag{9}$$

which can be used instead of the complicated 22 component, Eq. (8). This last equation (where we stopped explicitly showing the r-dependences of p and ρ, as well) could also have been derived from the local energy–momentum conservation equation, $T^{\mu\nu}_{;\nu} = 0$, which in turn follows from the mathematical fact $G^{\mu\nu}_{;\nu} = 0$ (the "contracted Bianchi identity") and the Einstein equation (1). This is the well-known statement that in GR, local energy–momentum conservation is built in.

Whichever set one chooses, {(6)–(8)}, or {(6), (7), (9)}; the EFE give three equations, but we have four unknown functions. Hence, another equation is needed to determine a definite solution. It is the approach to the choice of this extra equation that makes the