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Evaluation of free terms in hypersingular boundary integral equations

K. Davey *, A. Farooq

School of Mechanical, Aerospace and Civil Engineering, The University of Manchester, UK

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SUMMARY

The accurate numerical solution of hypersingular boundary integral equations necessitates the precise evaluation of free terms, which are required to counter discontinuous and often unbounded behaviour of hypersingular integrals at a boundary. The common approach for the evaluation of free terms involves integration over a portion of a spherical shaped surface centred at a singularity and allowing the radius of the sphere to tend to zero.

In this paper two alternative methods, which are shape invariant, are proposed and investigated for the determination of free terms. One approach, the *point-limiting method*, involves moving a singularity towards a shrinking integration domain at a faster rate than the domain shrinks. Issues surrounding the choice of approach and shrinkage rates, and path dependency are examined. A related approach, the *boundary-limiting method*, involves moving an invariant but shrinking boundary toward the singularity again at a faster rate than the shrinkage of the domain. The latter method can be viewed as a vanishing exclusion zone approach but the actual boundary shape is used for the boundary of the exclusion zone. Both these methods are shown to provide consistent answers and can be shown to be directly related to the result obtained by moving a singularity towards a boundary, i.e. by comparison with the direct method. Unlike the spherical approach the two methods involve integration over the actual boundary shape and consequently shape dependency is not a concern. A particular highlight of the point limiting approach, as a result of field approximations being restricted to the boundary, is the ability to obtain free terms in a mixed formulation without reference to the underpinning constitutive equations, which is not available to the spherical method.

Focus in the paper is on the 2-D potential equation as this is shown to be sufficient to demonstrate the concepts involved.

1. Introduction

Research into the development and application of hypersingular boundary integral equations has been ongoing over the past decade. The approach presents an alternative to the general solution of thermal, elastostatic and elastodynamic problems. However, it is invariably more costly computationally than the standard integral equation formulation involving kernels with greater complexity and higher-order singularities. The method is often employed in dual formulations in combination with standard integral methods; a common usage is fracture mechanics for the prediction of stress intensity and crack propagation [1–5].

The hypersingular approach is formulated as the sum of free terms and singular integrals incorporating two-point kernels. A number of investigations have postulated the existence of additional free terms [6–9] associated with corners connected to adjacent curved boundary parts. Free terms and associated integrals typically exist in the sense of the Hadamard finite part whose determination involves the creation of a vanishing exclusion zone and asymptotic analysis [10–16]. A reasonable review of the analytical treatments proposed for the evaluation of hypersingular integrals is given in Ref. [9].

The "natural" shape of the exclusion zone is a ε-ball $B_\varepsilon(\mathbf{x})=\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\|_2<\varepsilon\}$ whose boundary is an ε-sphere $S_\varepsilon(\mathbf{x})=\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\|_2=\varepsilon\}$ oriented by the normal vector $\mathbf{n}(\mathbf{y})$ pointing to the centre $\mathbf{x}$ [10]. However, it is recognised that the Hadamard finite part is not unique and depends on the shape of the vanishing exclusion zone [10,11]. A question that immediately arises: does the non-uniqueness lead to incorrect free terms in the governing integral equation? Shape dependency is a feature of the individual integrals in an integral equation and for these individual integrals the application of the spherical approach can differ with the result obtained using a direct approach [11]. However, despite these differences, integral equations contain combinations of said integrals which are shape independent giving rise to correct free terms. This paper re-examines the issue of shape dependency with the introduction of two new limiting processes for the evaluation of free terms in the hypersingular boundary integral equation for the

* Corresponding author.
E-mail address: keith.davey@manchester.ac.uk (K. Davey).

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potential problem. One method, denoted the *point-limiting method*, involves moving a singularity towards a shrinking integration domain at a faster rate than the domain shrinks. A particular concern and issue arising with this approach is path dependency. It is shown in the paper that individual integrals are path dependent in the sense that the angle at which a boundary is approached affects the limit. It is proved in this paper that path and shape dependency are intrinsically linked. Because no distortion of the boundary is involved a particular benefit of this approach is that boundary conditions can be directly incorporated into a free term. This option is not available to the spherical method which relies on constitutive relationships and the limiting process to arrive at the correct term. Although not a particular issue for the potential problem the use of constitutive equations in this way can be complex. It is worth highlighting at this point that boundary integral approaches can be viewed in the light of a strong variational method where different types of boundary condition can be approximated independently; a feature that is reinforced with the point-limiting approach.

A related approach, presented in the paper and denoted the *boundary-limiting method*, involves moving an invariant but shrinking boundary toward the singularity again at a faster rate than the shrinkage of the domain. The latter method can be viewed as a vanishing exclusion zone approach but the actual boundary shape is used for the boundary of the exclusion zone. Consistent results are achieved with the two methods.

In order to introduce the new subject matter, basic concepts are considered in Section 2 along with the potential hypersingular integral equation, which in this paper serves as a vehicle for illustrating the concepts involved. In Section 3, singularity annihilation, shape dependence, and the boundary and point-limiting approach are introduced. In Section 4 the theoretical aspects relating to path independence are examined in detail along with issues relating to the rate at which a boundary is approached. It is shown that shape and path independence are intrinsically linked as both properties stem from an integral identity of the form $\int_\Gamma g d\Gamma = 0$. In Sections 5–7 the point-limiting approach is applied to curved and planar boundaries, where it is established that analysis can be restricted to a local tangent plane for the least singular integral in the hypersingular equation. The limiting method is extended to a planar corner in Section 8 and a curved corner in Section 9. In Section 10, limiting exclusion zone methods are discussed along with the boundary-limiting approach. The free terms predicted by the various methods are compared, but also established in Section 10 is the equivalence of the boundary and point-limiting method. In the Section 11 the flexibility of the point limiting method is highlighted with its ability to cater for mixed boundary conditions without recourse to constitutive equations and thus particularly suited to the strong variational method. Finally in Section 12 a number of examples are considered to contrast the method against the direct approach.

2. Basic concept review

Consider the potential problem $\nabla^2 u = 0$ satisfied on the spatial domain Ω, where it is assumed that $u \in C^2(\Omega)$ (or $u \in C^\infty(\Omega)$). The boundary Γ for Ω is oriented by the outward pointing normal $\mathbf{n}$. Consider an arbitrary two-point function $w(\mathbf{x},\mathbf{y}) \in C^\infty$ for $\mathbf{x} \neq \mathbf{y}$ and application of Green's second identity gives

$$\int_\Omega w(\mathbf{x})\nabla^2 u d\Omega - \int_\Omega u \nabla^2 w(\mathbf{x}) d\Omega = \int_\Gamma w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma \quad (1)$$

where it is understood the that $d\Omega = d\Omega(\mathbf{y})$, $w(\mathbf{x}) = w(\mathbf{x},\mathbf{y})$, etc., with $\mathbf{y}$ excluded for convenience.

Consider the definition $\nabla^2 w = -\delta(\mathbf{x},\mathbf{y})$, where δ is the Dirac delta distribution satisfying $\delta(\mathbf{x},\mathbf{y}) = 0$ if $\mathbf{x} \neq \mathbf{y}$ and $\int_\Omega \delta(\mathbf{x},\mathbf{y}) d\Omega(\mathbf{y}) = 1$ if $\mathbf{x} \in \Omega$. In 2-D, a solution to this equation is Green's function $w = -(2\pi)^{-1} \ln r$, where $r = \|\mathbf{y} - \mathbf{x}\|_2$, which is evidently singular at $\mathbf{x} = \mathbf{y}$. The traditional approach for dealing with a singularity in the domain Ω is to exclude it by creating a vanishing exclusion zone Ω_ε with boundary Γ_ε. Recognising that Green's Theorem applies to multi-connected domains gives

$$\int_{\Omega-\Omega_\varepsilon} w(\mathbf{x})\nabla^2 u d\Omega - \int_{\Omega-\Omega_\varepsilon} u \nabla^2 w(\mathbf{x}) d\Omega = \int_\Gamma w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma + \int_{\Gamma_\varepsilon} w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma - \int_{\Gamma_\varepsilon} u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma \quad (2)$$

Since $\nabla^2 u = 0$ and $\nabla^2 w = 0$ on $\Omega - \Omega_\varepsilon$ it follows that:

$$\int_{\Gamma_\varepsilon} u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma - \int_{\Gamma_\varepsilon} w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma = \int_\Gamma w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma \quad (3)$$

which confers shape independency on the expression $w\partial u/\partial n - u\partial w/\partial n$ for potential functions w and u.

The issue of shape dependence is recognised not to be an issue for the integrals appearing in Eq. (3) since these are at worse Cauchy Principal Value singular. Note however that Eq. (3) is of the form $\int_{\Gamma\varepsilon} g d\Gamma = \int_\Gamma g d\Gamma$, where $g = u\partial w/\partial n - w\partial u/\partial n$ with $\underline{n}$ suitably redefined as $-\underline{n}$ on Γ_ε. This demonstrates that the limit $\lim_{\epsilon \to 0} \int_{\Gamma_\varepsilon} g d\Gamma$ does not depend on the shape of Γ_ε. This is an issue that is particularly important for hypersingular integral equations. Any free terms present are generated on the left-hand side of Eq. (3) in the limit $\varepsilon \to 0$. In taking this limit, knowledge about the behaviour of the function u and its normal derivative is required. It is common to assume for hypersingular integral equations that $u \in C^{1,\alpha}$ where $0 < \alpha \leq 1$, then $u(\mathbf{y}) = u(\mathbf{x}) + (y_i - x_i)\partial u/\partial y_i(\mathbf{x}) + O(r^\alpha)$ and $\partial u/\partial y_i(\mathbf{y}) = \partial u/\partial y_i(\mathbf{x}) + O(r^{1-\alpha})$ as $r \to 0$. It should be appreciated however that for internal points $u \in \Gamma_\varepsilon$ is simply a restriction denoted $u_{\Gamma\varepsilon}$. Thus $u_{\Gamma\varepsilon}$ takes on the properties of u in Ω, i.e. $u \in C^2(\Omega)$. In principle there is nothing preventing the repeated differentiation of Eq. (2) and it could be argued that $u \in C^\infty$ on Ω; a property transferred to $u_{\Gamma\varepsilon}$.

Since the shape of the Ω_ε is not an issue for the integrals on the left-hand side of Eq. (3); it is commonplace to set $\Omega_\varepsilon(\mathbf{x}) = B_\varepsilon(\mathbf{x}) = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\|_2 < \varepsilon\}$. Thus on setting $d\Gamma = \varepsilon d\theta$ gives

$$\int_{\Gamma_\varepsilon} u(\mathbf{y}) \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma(\mathbf{y}) = \int_0^{2\pi} (u(\mathbf{x}) + O(\varepsilon)) \frac{\partial}{\partial r}\left(\frac{\ln r}{2\pi}\right)\Big|_{r=\varepsilon} \varepsilon d\theta = u(\mathbf{x}) + O(\varepsilon) \quad (4)$$

and

$$\int_{\Gamma_\varepsilon} w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma = \int_0^{2\pi} \frac{\ln\varepsilon}{2\pi}\left(\frac{\partial u}{\partial r}(\mathbf{x}) + O(\varepsilon)\right)\varepsilon d\theta = O(\varepsilon \ln \varepsilon) \quad (5)$$

as $\varepsilon \to 0$.

Although it is usual to take the limit $\varepsilon \to 0$ prior to differentiation of Eq. (3) it is of interest to immediately differentiate Eq. (3) prior to taking the limit, i.e.

$$\int_{\Gamma_\varepsilon} u \frac{\partial w_i}{\partial n}(\mathbf{x}) d\Gamma - \int_{\Gamma_\varepsilon} w_i(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma = \int_\Gamma w_i(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w_i}{\partial n}(\mathbf{x}) d\Gamma \quad (6)$$

where

$$w_i = \frac{\partial w}{\partial x_i} = \frac{\partial r}{\partial x_i}\frac{dw}{dr} = \frac{\partial r}{\partial x_i}\frac{d}{dr}\left(-\frac{\ln r}{2\pi}\right) = \left(-\frac{y_i - x_i}{r}\right)\left(-\frac{1}{2\pi r}\right) = \frac{r_i}{2\pi r^2} \quad (7)$$

and

$$\frac{\partial w_i}{\partial n} = n_j \frac{\partial w_i}{\partial y_j} = n_j \frac{\partial}{\partial y_j}\left(\frac{r_i}{2\pi r^2}\right) = n_j \left(\frac{\delta_{ij} - 2r_i r_j / r^2}{2\pi r^2}\right) \quad (8)$$

where $r_i = y_i - x_i$.

Note that Eq. (6) immediately confers shape independency on the expression $w_i \partial u/\partial n - u\partial w_i/\partial n$ for potential functions w and u, since it is of the form $\int_{\Gamma\varepsilon} g_i d\Gamma = \int_\Gamma g_i d\Gamma$, where $g_i = w_i \partial u/\partial n - u\partial w_i/\partial n$ with