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More problems for Newtonian cosmology ☆

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ABSTRACT

I point out a radical indeterminism in potential-based formulations of Newtonian gravity once we drop the condition that the potential vanishes at infinity (as is necessary, and indeed celebrated, in cosmological applications). This indeterminism, which is well known in theoretical cosmology but has received little attention in foundational discussions, can be removed only by specifying boundary conditions at all instants of time, which undermines the theory's claim to be fully cosmological, i.e., to apply to the Universe as a whole. A recent alternative formulation of Newtonian gravity due to Saunders (Philosophy of Science 80 (2013) pp. 22-48) provides a conceptually satisfactory cosmology but fails to reproduce the Newtonian limit of general relativity in homogenous but anisotropic universes. I conclude that Newtonian gravity lacks a fully satisfactory cosmological formulation.

1. Introduction

Newtonian gravity in its original force-based formulation is defined for discretely many particles by

$$\ddot{x}^i_n(t) = G\sum_{m\neq n} m_m \frac{\left(x^i_m(t) - x^i_n(t)\right)}{\left|x_n(t) - x_m(t)\right|^3} \tag{1}$$

(m_n is the mass of the nth particle and x_n is its position vector at time t) or in the continuum limit by

$$\dot{v}^i(x, t) = G\int \mathrm{d}x'^3 \rho(x', t)\frac{x'^i - x^i}{|x' - x|^3} \tag{2}$$

where ρ is the mass density function, $v(x, t)$ is the velocity of a test particle at spacetime point x, t, and $\dot{v}(x, t)$ is the acceleration of that test particle, i.e. the time derivative along the particle's worldline.[1]

As has long been known (see Norton, 1999 for details of the history), this theory becomes ill-defined in cosmological applications, where the distribution of particles is homogeneous (i.e., if the mass density on large scales is asymptotically constant). For in that circumstance, (1)–(2) become conditionally convergent (the result of performing the sum depends on the order in which it is carried out) and thus ill-defined.

The point can be illustrated by elementary means (again, as has long been known). Consider a test particle a distance r from the central point O of a sphere of matter of uniform density ρ. The integral (2) can be split into two parts: one part that integrates over the matter in the smaller sphere of radius r centred on O, and another part that sums over a series of spherical shells of infinitesimal thickness, also centred on O and with radius $R > r$. It has been known since Newton that the force on a test particle external to a uniform sphere of matter is the same as would be the case if that matter were concentrated at the centre of the sphere, and that the force on a test particle inside a uniform spherical shell is zero. So the second part of that integral is zero, and the total acceleration of the particle is

$$\dot{v}^i(r) = -\frac{4}{3}\pi G\rho r\hat{r}^i = -\frac{4}{3}\pi G\rho r^i. \tag{3}$$

Allowing the radius of the sphere to increase without limit makes no difference to this argument, suggesting that (3) is also the correct acceleration for a test particle in an infinite medium of

☆Question marks (when not in their normal contexts) denote references removed for blind review.

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[1] Throughout this paper, Roman indices range from 1 to 3, and by definition $X_i = X^i$. x, without an index, is shorthand for the vector with coordinates x^1, x^2, x^3 where this is unambiguous; $|x| \equiv \sqrt{x^i x_i}$. (I adopt no abstract index notation and intend Roman-indexed objects to be the components of vectors and tensors in standard Cartesian coordinates.)

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density ρ. But in such a medium, the idea of a 'centre' becomes ill-defined, as the matter can be decomposed into concentric spheres around an arbitrary centre.

As has been repeatedly observed, however[2], this indeterminacy in the absolute acceleration has no actually observable consequences. For what is observable is (at most) the relative acceleration of two test particles, not the absolute acceleration relative to an unobservable background. And given two test particles with vector positions x_1^i, x_2^i, the relative acceleration is

$$\dot{v}^i(x_1) - \dot{v}^i(x_2) = -\frac{4}{3}\pi G\rho\left(x_1^i - x_2^i\right) \tag{4}$$

which depends on the separation of the particles but not on the location of the origin.

This suggests that in Newtonian cosmology properly formulated, only relative and not absolute accelerations matter — or put another way, that the split between inertial motion and acceleration under gravity is arbitrary. The most straightforward way to see this mathematically is to shift from the force-based to the potential-based formulation of Newtonian physics, in which the dynamics are given by

$$\dot{v}^i(x, t) + \nabla_i\Phi(x, t) = 0; \quad \nabla^2\Phi(x, t) = 4\pi G\rho(x, t) \tag{5}$$

(the latter being *Poisson's equation*). These equations are invariant under the transformations

$$\left(x^i, t\right) \to \left(x'^i, t\right) = \left(x^i + a^i(t), t\right) \tag{6}$$

$$\Phi \to \Phi'; \; \Phi'(x', t) = \Phi(x, t) - \ddot{a}_i(t)x^i + V(t). \tag{7}$$

where the $a_i(t)$ and $V(t)$ are arbitrary smooth functions of time. These are arbitrary time-dependent spatial translations; the transformation law for Φ generates a time-dependent but spatially independent acceleration that compensates for the effect of the translation.

In normal (i.e., non-cosmological) applications of Newtonian gravity, we impose a boundary condition

$$\lim_{|x|\to\infty}\Phi(x, t) = 0 \tag{8}$$

which serves to eliminate the gauge freedom in the equation for Φ, and to restrict the arbitrary translations $a_i(t)$ to include only time-independent translations and velocity boosts. But if we drop this assumption then Newtonian gravity becomes a theory in which only relative acceleration is well-defined, and absolute acceleration is pure gauge.

In this form, the theory lends itself naturally to a geometric interpretation, according to which the inertial structure of spacetime is not the absolute structure of Galilean spacetime (Anderson, 1967; Stein, 1967; Earman, 1970; Friedman, 1983) but determined locally by the matter distribution. This move is often accompanied by a differential-geometric reformulation of the theory— so-called *Newton-Cartan theory* — to replace the potential with a nonflat affine connection (see, e.g., Malament, 2012 and references therein). Knox (2014) argues that even in its standard potential formulation Newtonian gravity is already a theory of dynamically-determined inertial structure; Saunders (2013) goes further and argues that inertial structure can be entirely eliminated in Newtonian physics; in Wallace (2016a) I argue (from the currently-unfashionable coordinate-transform route to defining physical theories; cf Wallace (2016b)) that the transformation law (7) for Φ means that it is already a connection.

[2] See, e.g., McCrea and Milne (1934); Narlikar (1963); Davidson and Evans (1973); Evans (1974); Malament (1995); Norton (1995); Ellis and Dunsby (1997).

To see this further, consider a smooth distribution of test particles with velocities $v^i(x, t)$. The relative acceleration of infinitesimally close test particles is given by

$$\nabla_j\dot{v}^i(x, t) = \nabla_j\nabla^i\Phi(x, t) \tag{9}$$

and from a geometric perspective, this is a geodesic deviation equation and identifies the symmetric matrix $\Omega_i^{\,j} = \nabla_i\nabla^j\Phi$ as the (nontrivial part of the) spacetime curvature. Poisson's equation is now

$$\Omega_i^{\,i} = 4\pi G\rho \tag{10}$$

which may be thought of as a nonrelativistic version of the Einstein field equation $R_{\mu\nu} = 8\pi GT_{\mu\nu}$.

However, neither reconceptualising Newtonian gravity geometrically, nor reformulating it differential-geometrically, is required to apply it to cosmology: the simple potential-based form is already suitable (and the reformulations do not change the conclusions of this paper). In particular, the relative acceleration law (4) corresponds to a potential

$$\Phi(x) = \frac{2}{3}\pi G\rho|x|^2 + b_i x^i + V \tag{11}$$

which also satisfies $\nabla^2\Phi = 4\pi G\rho$ for spatially constant ρ. Translations now just serve to change the linear term in (11), which is equivalent to changing the centre around which the quadratic is defined, but have no effect on the relative accelerations.

So this seems to put Newtonian cosmology on a very satisfactory footing. And indeed, this approach to cosmology was developed by Heckmann and Schücking (1955, 1956), brought to wider attention by Szekeres and Rankin (1977) and is now the standard framework for cosmology where relativistic effects may be neglected (see, e.g., Ellis, 1971 for a comparison of relativistic and nonrelativistic cosmology); it also seems to have received widespread consensus in philosophy of physics (see, e.g., Malament, 1995; Norton, 1995; Pooley, 2013; Knox, 2014).

All seems well. The only trouble is: that boundary condition on Φ was there for a reason.

2. Non-uniqueness of solutions to poisson's equation

Let's review the normal route towards establishing equivalence of force-based and potential-based formulations of gravity (or indeed electrostatics). Given two solutions to Poisson's equation Φ_1, Φ_2, it follows that their difference $(\Phi_2 - \Phi_1)$ satisfies *Laplace's equation*,

$$\nabla^2(\Phi_2 - \Phi_1) = 0. \tag{12}$$

The operator ∇^2 is rotationally invariant and so elementary functional analysis (see, e.g., Jackson, 1999, p.95 for details) tells us that the most general solution to Laplace's equation defined over all space is

$$F(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^{\,m}|x|^l Y_m^l(\hat{x}) \tag{13}$$

for arbitrary constants $A_l^{\,m}$ (where the Y_m^l are spherical harmonics). So if Φ_1 and Φ_2 also satisfy the boundary condition (8) (so that $(\Phi_2 - \Phi_1)$ tends to zero at spatial infinity), we must have $\Phi_2 - \Phi_1 = 0$ everywhere, and so Poisson's equation has a unique solution. When we also observe that

$$K(x) = G\frac{1}{|x|} \tag{14}$$

satisfies