Engineering Analysis with Boundary Elements 34 (2010) 680–689

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# A meshless solution to two-dimensional convection–diffusion problems

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ARTICLE INFO

*Article history:*
Received 11 May 2009
Accepted 4 February 2010
Available online 19 March 2010

*Keywords:*
Meshless method
Integral equations
Circular sub-domains
Radial basis functions

ABSTRACT

An integral equation domain decomposition method has been implemented in a meshless fashion. The method exploits the advantage of placing the source point always in the centre of circular sub-domains in order to avoid singular or near-singular integrals. Three equations for two-dimensional (2D) or four for three-dimensional (3D) potential problems are required at each node. The first equation is the integral equation arising from the application of the Green's identities and the remaining equations are the derivatives of the first equation in respect to space coordinates. Radial basis function interpolation is applied in order to obtain the values of the field variable and partial derivatives at the boundary of the circular sub-domains, providing this way the boundary conditions for solution of the integral equations at the nodes (centres of circles). Dual reciprocity method (DRM) has been applied to convert the domain integrals into boundary integrals, though the approach is general and can be applied without the DRM. The accuracy and robustness of the method has been tested on a convection–diffusion problem. The results obtained using the current approach have been compared with previously reported results obtained using the finite element method (FEM), and the DRM multi-domain approach (DRM-MD) showing similar level of accuracy.



## 1. Introduction

Meshless integral equation approaches are receiving increased attention due to their accuracy associated with the integral equation methods, of which the most popular has been the boundary element method (BEM), and the flexibility they offer as the meshing requirements are either eliminated or reduced.

The local boundary integral equation (LBIE) was proposed by Zhu et al. and was applied to Poisson problems [1] and non-linear problems [2]. In LBIE the domain is sub-divided in a large number of sub-domains. In the work on LBIE reported so far, the support of the source point is taken to be a sub-domain in a shape of a circle though in theory the domain can be of any shape, with the source point in the centre of the circle. The most often used interpolation for field variables were the moving least-squares, though Sellountos and Sequeira [3] used augmented thin plate spline (ATPS) radial basis functions (RBFs) for interpolation of the field variable and gradients over the circular boundaries. The concept of "companion solution" is introduced in order to eliminate the single layer integral from the local boundary integral equation. In this way the potential field is the only unknown in the equations. For source points that are located on the (global) boundary of the given problem part of the local circular boundary is replaced by the part of the global boundary and the integrals are evaluated over this part of the global boundary and remaining part of the circle. For nodes on the boundary of the given problem the single layer integrals are present in the integral equation. This means that integration over the boundary must be performed which in a way departs from the meshless idea of the approach. The gradients of the potential field appear in the integrals over the boundary of the problem and they may either be obtained from the original BEM equations or they can be obtained by differentiating the interpolation function used to represent the potential over the local boundary.

Recently [3] LBIE has been employed for solution of the Navier–Stokes equations in combination with the radial basis functions (RBFs) used for interpolation of the field variables over the circular boundaries of the sub-domains.

Though the present formulation may seem similar to the LBIE in certain aspects, overall it is a fundamentally different approach. Similarly to the LBIE it is implemented over circular sub-domains where the source points are placed in the centres of the circles. The work follows the idea of Bui and Popov [4] who proposed using three equations at each source point for 2D problems solved using BEM with overlapping sub-domains. One equation is the original integral equation usually used in the direct formulation BEM, while the other two equations are the derivatives in respect to spatial coordinates of the original equation at the source point. In this work the augmented thin plate spline (ATPS) radial basis functions (RBFs) were used for interpolation of the field variable

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doi:10.1016/j.enganabound.2010.02.003

and gradients over the circular boundaries. This RBF was selected in order to use the same interpolation function for representing the field variables and for the approximation in the DRM part of the formulation. The DRM [5] is used to convert the domain integrals into integrals over the local circular boundaries. In the present case the Laplace fundamental solution is used to solve the convection–diffusion equation. The use of the ATPS in the DRM-MD has been elaborated elsewhere [6,7]. The present approach makes use of the DRM-MD since the results of this numerical scheme showed that this sub-domain BEM approach is very efficient for a number of different problems [8–14]. The current formulation is a natural transformation of the existing DRM-MD towards a meshless scheme by using the idea of Bui and Popov [4]. Since the proposed approach uses circular sub-domains, the classical integral equations from the BEM as well as the RBF interpolation for the field variables and the DRM part, further in this paper it will be referred to it as the radial basis integral equation method (RBIEM).

Though RBIEM seems similar to the LBIE because both use circular sub-domains, except on the global boundary where the LBIE does not use circular sub-domains, they are different since the LBIE uses the concept of "companion solution" in order to avoid solution for the gradients/normal derivatives inside the problem domain, while the RBIEM solves for the potential and partial derivatives at each node. This enables the RBIEM to be a truly meshless approach since the values of the normal derivatives are obtained everywhere including the source points located on the global boundary of the problem domain. The boundary conditions in the RBIEM are imposed directly at the source points on the global boundary. In the RBIEM there is no need for integration over any part of the global boundary of the problem domain. The integration is performed over the boundaries of the circular sub-domains, and since the source point is always in the centre of the circle, the integrals are regular, regardless of the order of the derivative, and easy to implement in the computer code. The matrix coefficients resulting from the integration over the circular boundaries will be the same and there will be no need to re-evaluate them for circles with the same radius.

Another important point is that the sub-domain BEM approaches like DRM-MD and boundary-domain integral method (BDIM) produce overdetermined systems of equations. The degree to which the system is overdetermined depends on the geometry of sub-domains used. This stems from the fact that there are matching conditions between sub-domains not only for potentials but for fluxes as well (more details can be found in [12]). While having continuity for potentials and fluxes is one of the advantages of the BEM sub-domain approaches in respect to formulations which preserve continuity for potentials only, e.g. the Galerkin FEM, the Green element method, having over-determined system of equations makes the solution process more complex. The RBIEM always produces a closed system of equations since the number of equations per node in the interior of the problem domain is three for 2D and four for 3D problems, unlike the DRM-MD and BDIM which produce higher number of equations in the nodes in the interior which are shared by several sub-domains.

In the current case the RBIEM has been implemented with the DRM, although in principle it can be implemented with different numerical schemes, including schemes where the domain integrals are directly evaluated. This approach is especially effective in applications where the partial derivatives in respect to coordinates are required, e.g. the convection–diffusion equation and the Navier–Stokes equation.

Further in the paper the "global boundary" will mean the boundary of the given problem and the "local (circular) boundary" will mean the boundary of the circular sub-domains.

## 2. The boundary element dual reciprocity method

In this section, the boundary element dual reciprocity method (DRM) is introduced though the proposed approach will be effective with other BEM formulations as well. Let us consider the following equation:

$$\nabla u^2(r) = b\left(r, u(r), \frac{\partial u(r)}{\partial x_i}, \frac{\partial u(r)}{\partial t}\right) \tag{1}$$

where $u(r)$ is the potential field, $r$ the position vector, $x_i$ the component of $r$ and $t$ the time. Given a point $r$ inside a domain $\Omega$, by applying the Green integral formula Eq. (1) within the domain $\Omega$ bounded by the boundary $\Gamma$ can be transformed into the following integral form:

$$\lambda(r)u(r) + \int_\Gamma q^*(r,\xi)u(\xi)\,d\Gamma_\xi - \int_\Gamma u^*(r,\xi)q(\xi)\,d\Gamma_\xi = \int_\Omega u^*(r,\xi)b(\xi)\,d\Omega_\xi \tag{2}$$

Here $u^*(r,\xi)$ is the fundamental solution of the Laplace problem which is given by

$$u^*(r,\xi) = \frac{1}{2\pi}\ln\left(\frac{1}{R}\right) \tag{3}$$

for a 2D problem and

$$u^*(r,\xi) = \frac{1}{4\pi}\frac{1}{R} \tag{4}$$

for a 3D problem, where $R$ is the distance from the point of application of the concentrated unit source to any other point under consideration, i.e. $R=|r-\xi|$ and $q(\xi)=\partial u(\xi)/\partial n$ and $q^*(r,\xi)=\partial u^*(r,\xi)/\partial n$. The constant $\lambda(r)$ has value from 0 to 1 being equal to 1/2 for smooth boundaries and 1 if the source point $r$ is inside the domain.

The DRM approximation [5] is introduced to transform the domain integral in (2) in terms of equivalent boundary integrals. The basic idea is to expand the $b(\xi)$ term using approximation functions, i.e.:

$$b(\xi) \cong b = \sum_{k=1}^{J_{bn}+J_{in}} \alpha_k f(\xi, \eta_k) \tag{5}$$

where $f(\xi,\eta_k)$ is the approximation function which depends on the location of nodes $\eta_k$, and $\alpha_k$ are unknown coefficients. The approximation employs $J_{bn}$ nodes on the boundary and $J_{in}$ nodes inside the domain.

The most popular choice in the past for the function $f(\xi,\eta_k)$ in the DRM approach was [15]

$$f(\xi, \eta_k) = 1 + R_{DRM}(\xi, \eta_k) \tag{6}$$

where $R_{DRM}(\xi,\eta_k)$ is the distance between a pre-specified fixed collocation point $\eta_k$ and a field point $\xi$ where the function is approximated, i.e. $R_{DRM}(\xi,\eta_k)=|\xi-\eta_k|$.

The function $f$ given in (6) belongs to a family of functions called radial basis functions (RBFs), which relate to the multi-variate function [16]. Micchelli [17] proved that for some additional conditions and when the nodal points are all distinct, the matrix resulting from a RBF interpolation is always non-singular. Therefore, as long as the function $b$ in (1) is regular the coefficients $\alpha_k$ are well defined. The use of different types of RBFs in DRM was discussed by Golberg and Chen [18–20].

With the DRM approximation for the non-homogeneous term $b$ the domain integral in Eq. (2) becomes

$$\int_\Omega u^*(r,\xi)b(\xi)\,d\Omega_\xi \cong \sum_{k=1}^{J_{bn}+J_{in}} \alpha_k \int_\Omega u^*(r,\xi)f(\xi,\eta_k)\,d\Omega_\xi \tag{7}$$