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Local integral equations implemented by MLS-approximation and analytical integrations

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ABSTRACT

This paper is devoted to the development of advanced mesh free implementations of the governing equations and the boundary conditions for boundary value problems in elasticity. Both the strong and weak formulations are discretized by using the Moving Least Squares approximations. The weak formulation is represented by local integral equations considered on sub-domains around interior nodal points. The awkward evaluation of the shape functions and their derivatives is reduced by focusing to nodal points because of the development of analytical integrations. That results in significant saving of the computational time needed for creation of the system matrix. Furthermore, a modified differentiation scheme is developed for approximation of higher order derivatives of displacements appearing in the discretized formulations. The accuracy, convergence and computational efficiency are studied in simple numerical example.

1. Introduction

Recently a great attention has been paid to the development of various mesh free formulations for solution of boundary value problems in many branches of science and engineering. All of such formulations utilize mesh free approximations of field variables, but not all of them result in truly mesh free formulations. In contrast to the strong and/or weak local formulations, the global weak formulations still need the background mesh for integration. Atluri and co-workers remarked that local character of the formulations known as Meshless Local Petrov–Galerkin Method gives rise to the development of truly mesh free formulations (see e.g. [1] from the huge list of literature about MLPG).

The importance of mesh free formulations is justified especially in problems with moving boundaries and/or in separable media where remeshing is required in mesh based approaches. Besides the savings due to avoiding mesh generation, there are also other advantages of mesh free formulations as compared with element based methods. For instance, the problem of discontinuities on element junctions disappears; the difficulties with searching the fundamental solutions and/or evaluation of singular integrals in BEM formulations are avoided; the mesh free formulations give the possibility of an elegant treatment of material in-homogeneity in functionally graded materials without increasing the computational effort.

Recently, a renaissance and development of many mesh-free approximations appeared in literature. We name only two of them, namely the Point Interpolation Method (PIM) and the Moving Least Squares (MLS) approximation (see e.g. [2–4]). Anyway, the discrete degrees of freedom are associated with nodal points, which are spread both in the interior of the analyzed domain and on its boundary. Thus, mesh free approximations are domain-type approximation for primary field (displacements). Hence, differentiating the approximation of displacements, one can obtain also the approximation of gradients of displacements. Then, the numerical implementation of prescribed boundary conditions is available. In order to obey physically meaningful interaction among the discrete d.o.f. and satisfy spreading of the influence throughout the body, correct coupling relationships for nodal unknowns should be derived. Since the governing equations are to be satisfied at each point and/or balance equations on an arbitrary sub-domain of the whole body, one can utilize them in

Abbreviations: FGM, functionally graded materials; LIE, local integral equations; MLS, Moving Least Squares; MLS-stand, standard MLS-approximation approach; MLS-CAN, Moving Least Squares approximation with Central Approximation Node concept; CPDE, collocation of partial differential equations; *sdif*, standard differentiation approach; *mdif*, modified differentiation approach; LIE-TSE(YMDG)-sdif, computational technique based on LIE, Taylor series expansion (of Young Modulus, displacement gradients) with using *sdif*-approach; LIE(ni), computational technique based on LIE with numerical integration; LIE(ni), YM ex., DG-sdif, computational technique based on LIE(ni) with exact values for the Young Modulus and using *sdif* for displacement gradients; DG-mdif, computational technique utilizing *mdif* for displacement gradients; DG-sdif ($sfdo=2$), computational technique utilizing DG-sdif with maximal shape function differentiation order being equal to 2; LIE(ai), YM(6), computational technique based on LIE with analytical integration and using Taylor series expansion for the Young Modulus up to 6th order

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the derivation of certain integral relationships on arbitrary finite parts of the body in contrast to the global integral formulation.

Recall that the most familiar domain-type approximation is represented by finite elements [5]. Eventually, one can utilize the finite element approximation for the numerical implementation of local integral equations (derived as integral form of balance equations on sub-domains) considered on appropriately chosen sub-domains as unions of several neighbouring elements [6–9]. In such implementations, one avoids the weak point of mesh-free approximations consisting in computational inefficiency due to relatively complicated and tedious evaluation of shape functions and their derivatives. Despite the possibility to utilize finite elements, the advantages of mesh free approximations are worth eliminating the mentioned handicap. Certain simplification in computation of the shape function derivatives has been achieved by expressing them in terms of the first order derivatives [13–15].

In this paper, we present development of advanced mesh free implementations of the governing equations and the boundary conditions for boundary value problems in elastostatics. The governing equations are considered either in the strong form as the partial differential equations for displacements or in the weak form as local integral equations (LIE) derived from the force equilibrium on sub-domains around interior nodal points. For mesh free approximations, we have utilized both the standard Moving Least Squares (MLS) approximation and the MLS combined with the Central Approximation Node (CAN) concept [10,11]. The prolongation of the computational time due to awkward evaluation of the shape functions and their derivatives can be diminished by combining meshless approximations with analytical integrations. Besides the LIE formulation also the collocation of the governing partial differential equations is considered (CPDE). Furthermore, a modified differentiation scheme is developed for approximation of higher order derivatives of displacements appearing in the discretized formulations [14,15]. A simple example is considered in numerical computations with using the analytical solution as benchmark solution in accuracy and convergence study. The computational efficiency is assessed by the time needed for creation of the system matrix.

2. Linear elasticity problems in continuously non-homogeneous media. Governing equations – strong and weak formulations

The general physical balance principles of a continuum theory take the form of integral equations. The governing equations or field equations are given by differential equations, which can be derived from these integral equations by taking into account that they hold for all arbitrary but small material domains. The governing equation for linear elasticity problems in anisotropic and continuously non-homogeneous media is given by the following partial differential equation with variable coefficients [12]:

$$(c_{ijkl}(\mathbf{x})u_{k,l}(\mathbf{x}))_{,j} = -X_i(\mathbf{x}), \text{ in } \Omega \tag{1}$$

where $u_i(\mathbf{x})$ is the unknown displacement field, $X_i(\mathbf{x})$ is the known density of body forces, and $c_{ijkl}(\mathbf{x})$ describe the spatially dependent material coefficients. The differential form of the governing equations supplemented with prescribed boundary conditions is known as the strong formulation of a boundary value problem.

The most natural weak formulation can be obtained by replacing the differential governing equations by their generator, i.e., by integral form of the force equilibrium considered on arbitrary material sub-domains. The derivation of the integral form of the governing equations from the differential ones can be done easily by reversing the derivation of differential governing equations from the integral form of the force equilibrium. Really, bearing in mind the Hooke's law for linear elastic, anisotropic and non-homogeneous continuous media

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x})u_{k,l}(\mathbf{x}) \tag{2}$$

with $\sigma_{ij}(\mathbf{x})$ being the stress tensor components, one can rewrite Eq. (1) as

$$\sigma_{ij,j}(\mathbf{x}) + X_i(\mathbf{x}) = 0 \tag{3}$$

and hence, integrating Eq. (3) over an arbitrary material domain $\omega \subset \Omega$, one obtains the integral relationships

$$\int_{\omega} [\sigma_{ij,j}(\mathbf{x}) + X_i(\mathbf{x})] d\omega(\mathbf{x}) = 0 \tag{4}$$

which is equivalent with the integral form of the force equilibrium, since the domain integral of the first term in Eq. (4) can be rewritten in view of the Gauss divergence theorem into the boundary integral of surface forces and Eq. (4) becomes

$$\int_{\partial\omega} t_i(\boldsymbol{\eta}) d\Gamma(\boldsymbol{\eta}) + \int_{\omega} X_i(\mathbf{x}) d\omega(\mathbf{x}) = 0 \tag{5}$$

where the surface tractions $t_i(\boldsymbol{\eta})$ are expressed in terms of the stress tensor components and unit outward normal vector $n_j(\boldsymbol{\eta})$ as

$$t_i(\boldsymbol{\eta}) = n_j(\boldsymbol{\eta})\sigma_{ij}(\boldsymbol{\eta}) \tag{6}$$

In what follows, we shall consider isotropic medium, when the number of material coefficients is reduced to two which can be selected as the Young modulus $E(\mathbf{x})$ and the Poisson ratio $\nu(\mathbf{x})$. Then

$$c_{ijkl}(\mathbf{x}) = E(\mathbf{x})c^{o}_{ijkl}(\mathbf{x}),\ c^{o}_{ijkl}(\mathbf{x}) = \frac{1}{2(1+\nu)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right) \tag{7}$$

Usually, the Poisson ratio is constant.

Finally, the prescribed boundary conditions (b.c.) of the elasticity problem can be of the following types:

$$\begin{aligned} &\text{essential (or Dirichlet) b.c.}: \ u_i(\boldsymbol{\eta}) = \overline{u}_i(\boldsymbol{\eta}) \text{ at } \boldsymbol{\eta} \in \partial\Omega_D \\ &\text{natural (or Neumann) b.c.}: \ t_i(\boldsymbol{\eta}) = \bar{t}_i(\boldsymbol{\eta}) \text{ at } \boldsymbol{\eta} \in \partial\Omega_N \end{aligned} \tag{8}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, and an over-bar denotes the prescribed quantities.

Substituting (7) into (1), the governing equation in the strong formulation becomes

$$E(\mathbf{x})c^{o}_{ijkl}u_{k,lj}(\mathbf{x}) + E_{,j}(\mathbf{x})c^{o}_{ijkl}u_{k,l}(\mathbf{x}) = -X_i(\mathbf{x}) \tag{9}$$

Note that the strong formulation works with displacements and their first as well as second order derivatives of displacements, while the weak formulation involves only displacements and their gradients.

In discretization of the strong formulation, a domain-type approximation of displacements (the dimension of the approximation domain as well as influence domain are identical with that of the analyzed domain) is needed. Then, the approximate expressions for the derivatives of displacements can be obtained by differentiating the approximations of displacements. Eventually, both the governing equations and the boundary conditions can be discretized and solved for discrete unknowns. Such an approach is applicable also to the weak formulation of the governing equation, where we need only the gradients of displacements.

Moreover, it is well known that a pure boundary integral formulation does not exist in the case of general continuously non-homogeneous media because of absence of fundamental solutions. Thus, utilization of interior unknowns or certain kind of domain-type approximation seems to be necessary.