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Novel applications of BEM based Poisson level set approach

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ABSTRACT

Accurate and efficient computation of the distance function d for a given domain is important for many areas of numerical modeling. Partial differential (e.g. Hamilton–Jacobi type) equation based distance function algorithms have desirable computational efficiency and accuracy. In this study, as an alternative, a Poisson equation based level set (distance function) is considered and solved using the meshless *boundary element method* (BEM). The application of this for shape topology analysis, including the medial axis for domain decomposition, geometric de-featuring and other aspects of numerical modeling is assessed.

1. Introduction

Distance level set, d, is a key parameter in many (numerical) simulation approaches [1], in peripheral applications incorporating additional solution physics [2] and also in mesh generation [3]. It has proved important in computer vision, solid modeling and other computational science. Especially, distance function can be helpful in constructing the *medial axis transform* (MAT) for a given geometry [4]. The latter is regarded as a key step in shape analysis and solid modeling [5,6].

In this paper, as a further study to [4], we propose a hybrid MAT approach based on the Poisson distance function, namely d_P-MAT. The advantage of such a hybrid method being to extract a well approximated medial axis point cloud on a properly calculated distance field has been discussed in detail in Refs. [4,7]. A simple Laplacian or Hessian determinant criterion of the distance field can be applied to mark the medial axis, and optionally combined with the α-shape thinning/representation techniques to thin the marked area to mathematically thin curves. Results show that the Poisson distance is very competitive in predicting wall proximity and at the same time efficient.

Another application of d_P is for geometric de-featuring. In *computer aided design* (CAD), the model of a part is often used for analysis. These aesthetically pleasing models contain much manufacturing information that is superfluous to analysis requirements. It would be highly advantageous to easily alter the CAD model to suit analysis purposes. De-featuring, or feature suppression, involves removing small, detailed information from the CAD model. These small features increase the density of the mesh, thus increasing the complexity of both the mesh generation and simulation, without adding significant information to the solution. Also, it is an extremely time-consuming and difficult process to remove features by hand. There are a number of methods so far seen in the literature [8–10]. The crucial part here is the automation. The possibility of using distances for automatic geometrically shape feature removing is explored here with the Poisson distance function.

In this study, the focus is on the solution of a Poisson distance function equation within the framework of the meshless boundary element methods. First, we will discuss the Poisson equation of an auxiliary scalar $\nabla^2\psi=\rho$, where ψ is linked to the distance function d by a quadratic equation. The attraction of applying BEM here is driven by efficiency. Without meshing and integrating in the interior elements, the computation can usually be reduced by one dimension. That is to say, if the original distance problem is fully 3D, with BEM we only need to solve a 2D problem. It is also advantageous in moving boundary problems, where other discretization methods would require the entire interior domain to be re-meshed.

However, as noted by Ramsak and Skerget in their recent work [11], there are not many efficient BEM formulations for a generic Poisson equation in the literature. For example, Suciu et al. [12] used the Galerkin vector approach but limited the source function ρ to satisfy $\nabla^2\rho=const$ rather than zero, i.e. harmonic. Fortunately, as will be shown later, in the present study the auxiliary Poisson equation of ψ in the distance function context can be actually reduced to a Laplacian equation due to the fact that the inhomogeneous term ρ can be chosen as a constant without affecting the distance solution.

This paper is organized as follows. The Poisson equation of ψ is introduced in the next section followed by the numerical solutions and test cases. The d_P-MAT approach for medial axes are

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discussed in Section 4. Finally, the application of distance functions for geometric de-featuring is discussed.

2. Derivation of Poisson distance function

In contrast to the eikonal exact distance governing equation, other differential (and indeed integral) equations describing the 'distance' function are also possible. Several of these methods were specifically sought to find the wall proximity for turbulence modeling [13]. One of these approaches goes back to the work of Spalding [14], in which the Poisson equation of an auxiliary variable ψ is solved.

Consider the ultimately simple case, 1D distance from the center point, and suppose there is a scalar function $\psi(x)$, such that

$$\frac{d^2\psi}{dx^2} = -1 \quad \text{and} \quad \psi(0) = 0 \tag{1}$$

After integration twice, it becomes

$$\frac{d\psi}{dx} + x - C_0 = 0 \quad \text{and} \quad \psi + \frac{x^2}{2} - C_0 x + C_1 = 0 \tag{2}$$

Combining the above and the boundary condition in (1), we can arrive at the quadratic equation of x:

$$\frac{1}{2}x^2 + \frac{d\psi}{dx}x - \psi = 0 \tag{3}$$

and the solution x, i.e. the distance function d, can be expressed as

$$d = x = -\frac{d\psi}{dx} \pm \sqrt{\left(\frac{d\psi}{dx}\right)^2 + 2\psi} \tag{4}$$

where the positive solution is meaningful. It then can be extended to multi-dimension with $d\psi/dx$ replaced by $\partial\psi/\partial n$ where n is the outward unit normal, and the governing equation for $\psi(\mathbf{x})$ becomes

$$\begin{cases} \nabla^2\psi = \rho, & \mathbf{x} \in \Omega \\ \psi = 0, & \mathbf{x} \in \partial\Omega \end{cases} \tag{5}$$

where the source function $\rho = -1$, and the distance can be approximated as

$$d \approx n = -\left|\frac{\partial\psi}{\partial n}\right| \pm \sqrt{\left(\frac{\partial\psi}{\partial n}\right)^2 + 2\psi} \tag{6}$$

Although Eqs. (1) and (4) are exact, the extension (6) to multi-dimension is not. However, for most numerical modeling problems considered here, only the near wall accuracy matters. Another advantage of Eqs. (5) and (6) is that they overestimate d around sharp convex surfaces and underestimate it around concave. This in exactness has consistent traits with the solid angle based integral equation for distance function equation proposed by Launder et al. [13] and Spalart [15] intended for improving the modeling of turbulence model physics and potentially other aspects of numerical modeling.

3. Numerical solution methods

3.1. Fast multipole BEM solution

The solution of the Poisson equation of the domain integral has been formulated using the Galerkin vector approach. It requires the knowledge of a fundamental solution and is valid only for harmonic source terms [12]. Fundamental solutions are known for some, but not all, differential equations. Other methods such as the *dual reciprocity method* (DRM) [16] and the *multiple reciprocity method* (MRM) [17] may be used for more complex problems. However, for the current special case, where $\rho = const$, the Poisson equation (5) can be reduced to a Laplacian equation and solved using the efficient *fast multipole method* (FMM) by Liu and Nishimura [18].

Let ϕ be a new function, such that

$$\phi = \psi + f(x,y) \tag{7}$$

Thus Eq. (5) becomes

$$\begin{cases} \nabla^2\phi = 0, & (x,y) \in \Omega \\ \phi = f(x,y), & (x,y) \in \partial\Omega \end{cases} \tag{8}$$

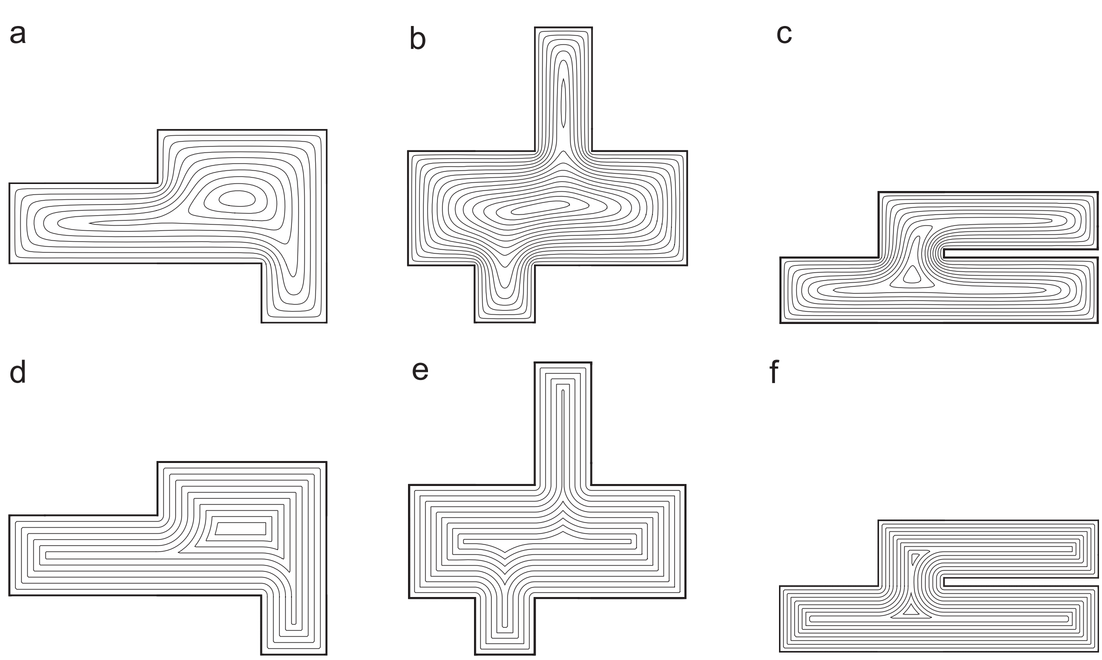

Fig. 1. The 2D d contours for three domains. *Upper*: Poisson. *Lower*: exact.