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Development and implementation of some BEM variants—A critical review

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ABSTRACT

Due to rapid development of boundary element method (BEM), this article explores the evolution of BEM over the past half century. We here summarize the overall development and implementation of several well-known BEM variants that includes collocation BEM, galerkin BEM, dual reciprocity BEM, complex variable BEM and analog equation method. Their theoretical and mathematical backgrounds are carefully described and a generalized Laplace's equation (and Poisson's equation) is utilized in demonstrating the different approaches involved. An up-to-date review on characteristics and implementation for each of the five variants is presented and also highlighted their significant contributions in boundary element research. In addition, this article tries to cover whole aspect of interests including efficiency, applicability and accuracy in order to give better understanding of BEM evolution. Comparisons and techniques of improvement for these variants are also discussed.

1. Introduction

Boundary element method (BEM) is one of the numerical techniques designed for solving boundary value problems in partial differential equations (PDEs). Applied in various engineering and science disciplines, BEM can be considered as a major numerical method alongside with better known finite element method (FEM) and finite difference method (FDM) jointly providing effective computational solution for a wide class of engineering and scientific problems. Like many other numerical techniques, the interest towards BEM increased gradually over recent decades catalyzed by rapid advances in computer technology. In fact, the statistical data based on the *Web of Science* search shows that the amount of annual published literature described by BEM saw an exponential growth until late 1990s and exceeded 700 literatures annually in subsequent years [39]. In this relatively short period of time, the evolution of BEM is tremendous. Several comprehensive reviews on BEM recently prepared by scholars have documented various theoretical basis and early development of BEM [39,50,106,256]. Many of the mathematical approaches presented in BEM are associated with work of famous mathematicians and scientists. The contributions of mathematicians like Laplace, Green, Fredholm, Fourier, Kellogg and Betti could be traced in the theoretical and mathematical foundation of boundary integral equation (BIE) in the early 20th century (See [39] for more detailed description on their contributions). However, pioneering work by Jaswon [113] and Symm [243] in 1963 has been marked as the formal beginning of the boundary element era. They demonstrated direct formulation using Green's third identity for two-dimensional Laplace's equation, hence, credited them as the first to formulate the potential problem in terms of direct BIE that set them apart from existing indirect BIE. Although, in the beginning, their work did not receive much attention as they deserved, their concept on direct approach has inspired Rizzo [219] to adopt similar approach using displacements and tractions in integral equation. Soon afterwards, his work has attracted and inspired researchers to investigate the potential of the new boundary integral approach. Considered as a major breakthrough in BEM, his work is then adopted by many researchers and saw swift progress in the development of boundary integral equation approach.

Following these early works, extensive researches and development works were carried out in 1970s to construct the basis of modern BEM on various aspects including basic principles, applications and numerical techniques. One must bear in mind that before BEM existed, Green's formulation used only to reduce the differential equation in the domain-to-boundary integral equation. Only until late 1970s, the discretization of these boundary integrals using numerical techniques creates so-called BEMs. The terminology of boundary element method, previously referred as boundary integral equation method or boundary integral method first appeared in the works of Brebbia and Dominguez [28] and Banerjee and Butterfield [18] in 1977. However, the article prepared by Banerjee and Butterfield was still based on indirect method boundary equations while Brebbia and Dominguez presented the potential problems through a weighted residual approach using direct version. A year later, Brebbia [25] published the first textbook of BEM focusing on basic

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principles and its application to potential and elasticity problems, created a major advance in BEM. At the same time, enormous research efforts have been directed in expanding the development of BEM. One of the significant achievements was the implementation of substructure technique in BEM presented by Lachat [143,144] which overcome one of the major drawbacks in BEM: nonsymmetrical matrix. On the other hand, apart from conventional BEM in frequency domain, the time domain BEM was later introduced by Cole et al. [46] for anti-plane strain problem in two-dimensional elastodynamics and then improved by Mansur and Brebbia [153,154] to accommodate scalar wave problem. Following the extension of this formulation, the BEM can be employed to investigate the transient behavior and also nonlinear problems. The detailed theoretical basis for these problems (nonlinear and time-dependent problems) can be obtained in textbooks written by Brebbia and his colleagues in [30,31]. Soon, the BEM was further extended to cover a wide scope of solid mechanics [6,249] and fluid mechanics [32] problems. Thanks to early development prepared by numerous researchers including Brebbia and his fellow colleagues who directly involved in constructing the blueprint of modern BEM which then well received by the scientific community that nowadays the BEM has been adopted by many researchers covering countless specialized areas such as acoustics [8,45,253,262], contact mechanics [152], dynamics analysis [61], solids and structures [9,231], soil-structure interactions [84], nonlinear fluid dynamics [20], heat transfer [262,263] and quantum mechanics [215]. With the solid foundation and rich heritage, BEM emerged as a powerful method and thus become a strong alternative to the FEM and FDM.

2. Boundary integral equation

Unlike FEM and FDM, as the numerical implementation of boundary integral equations, BEM requires surface-only discretization. The BIE re-formulations of boundary value problems for partial differential equations are valid everywhere—interior and exterior of the domain and also on the boundary, giving it a big advantage since most of the engineering and scientific applications can be described by PDEs. Note, however, that not all PDEs can be transformed into integral equations, and therefore, it is crucial for researchers to be able to understand the classification of PDEs and the concept behind each class. In most mathematics books, these PDEs can be classified as being elliptic, parabolic, or hyperbolic type according to the form of the equation. In second-order PDE (the order of a PDE indicates the order of the highest order derivative found in the PDE), the differential equation is considered as: (1) elliptic equation when the coefficients of both no mixed second-order derivatives are nonvanishing and of the same sign. (2) parabolic equation when only one second-order non-mixed derivative term is present and of opposite sign. (3) hyperbolic equation when the coefficients of the non-mixed two second-order derivatives are nonvanishing and opposite in sign. Moreover, elliptic PDEs have boundary conditions specified around a closed boundary, whilst hyperbolic and parabolic PDEs have at least one open boundary. Thus, elliptic equations are often used to describe systems in the equilibrium or steady state whereas the parabolic equations being utilized for demonstrating physical systems with a time variable, diffusion like phenomena, and for the hyperbolic equations, they are frequently used to describe oscillatory systems especially wave-like phenomena. In addition to the classification mentioned above, PDEs can be presented as single PDE or systems of PDEs with multiple variables. Although the integral equation re-formulation can only be derived for certain classes of PDE, it is much easier to apply and more computationally efficient, if applicable. Therefore, it is critically important to know the characteristics of each differential equation in order to employ the right numerical technique. To date, the integral equations have been successfully applied to describe various physical disciplines such as elastostatics, electrostatics, electrodynamics, elasticity, plasticity, heat transfer, acoustics, fluid dynamics and so on. Table 1 illustrates some applicable areas for linear and nonlinear problems distinguished into classes of PDEs.

Table 1
Applicable areas for different classes of PDEs.

Linear	Nonlinear
Single second-order PDEs	
(A) Elliptic	(A) Hyperbolic
Laplace's equation	*Burgers' equation*
• Electrostatics	• Fluid mechanics
• Incompressible potential flow	
• Heat Conduction (Steady state, no heat generation)	
Poisson's equation	
• Electrostatics	
• Electromagnetics	
• Heat Conduction (Uniform thermal conductivity, steady state, heat generation within the solid)	
Helmholtz equation	
• Acoustics (Interior and exterior problem)	
• Electromagnetics	
• Optics	
(B) Parabolic	
Diffusion/Heat equation	
• Heat Conduction (Uniform thermal conductivity, no heat sources)	
Schrödinger's equation	
• Quantum mechanics	
(C) Hyperbolic	
Wave equation	
• Acoustics	
• Electromagnetics	
• Fluid dynamics	
Single fourth-order PDEs	
Biharmonic equation	
• Thin plate problem	
• Fluid-flow problem	
System of PDEs	
Navier's equation	*Navier–Stokes equation*
• Elasticity problem	• Fluid dynamics
Maxwell's equation	*Euler's equation*
• Electromagnetics	• Fluid dynamics