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A dual-reciprocity boundary element approach for axisymmetric nonlinear time-dependent heat conduction in a nonhomogeneous solid

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ABSTRACT

A dual-reciprocity boundary element method is presented for the numerical solution of a nonsteady axisymmetric heat conduction problem involving a nonhomogeneous solid with temperature dependent properties. It is applied to solve some specific problems including one which involves the laser heating of a cylindrical solid.

1. Introduction

Thermal conductivity and specific heat capacity of metallic solids have been experimentally observed to be strongly dependent on temperature during processes such as metal quenching. Thus, the development of numerical techniques for nonlinear heat conduction in solids with temperature dependent material properties has attracted the attention of many researchers in computational heat transfer. Earlier works on boundary element methods, such as Kikuta et al. [12] and Goto and Suzuki [9], assume that the solids are thermally isotropic and have density, specific heat capacity and thermal conductivity which are functions of temperature alone. Clements and Budhi [8], Azis and Clements [5] and, more recently, Ang and Clements [2] have proposed boundary element procedures for thermally anisotropic solids with material properties that vary with temperature and spatial coordinates.

The works in [2,5] are also applicable to linear heat conduction in nonhomogeneous media such as functionally graded materials. The analysis of functionally graded materials is a topic of special interest in boundary element methods. Some papers on boundary element methods for solving linear problems involving nonhomogeneous media with properties that vary continuously in space include Ang et al. [3], Clements [7], Kassab and Divo [11], Park and Ang [16], Rangogni [17], Tanaka et al. [18] and other relevant references therein.

The present paper considers a nonlinear time-dependent axisymmetric heat conduction problem involving a nonhomogeneous thermally isotropic solid with temperature dependent properties. Such a problem is of practical interest as axisymmetric structures can be found in many engineering applications (such as pressure vessels and piping components). The analyses in Ang and Clements [2], Azis and Clements [5] and Brebbia et al. [6] are used as a guide to convert the nonlinear partial differential equation governing the axisymmetric heat conduction into a suitable integro-differential equation. In addition to a boundary integral over a curve on an appropriate coordinate plane, the integro-differential equation also contains a domain integral. The dual-reciprocity approach is used here to express the domain integral approximately in terms of line integrals. The time derivative of the temperature in the integro-differential formulation is approximated using a finite difference formula. At any given time level, if the temperature is assumed known at earlier time levels, the problem under consideration is formulated in terms of a system of nonlinear algebraic equations to be solved using a predictor–corrector (iterative) procedure.

The numerical procedure presented here is applied to solve some specific problems including one which involves the laser heating of a cylindrical solid. For problems which have known exact solutions, the accuracy of the numerical solutions obtained is assessed.

2. The problem

Consider a thermally isotropic solid occupying the three-dimensional region R. If T is the temperature inside the solid, then the conservation of energy and the classical Fourier's law of heat conduction require the temperature to satisfy the partial differential equation

$$\nabla\cdot(\kappa\nabla T)+Q=\rho c\frac{\partial T}{\partial t}\quad \text{in } R \text{ for } t\geq 0, \tag{1}$$

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where ∇ is the gradient (nabla) operator, $\cdot$ denotes the dot product, t is time, ρ, c and κ are, respectively, the density, specific heat capacity and thermal conductivity of the solid and Q is the internal heat source generation rate.

With reference to a Cartesian coordinate system denoted by $Oxyz$, the geometry of the region R is symmetrical about the z-axis, that is, the boundary of R can be obtained by rotating a curve on the Oxz plane by an angle of 360° about the z-axis. Furthermore, if r and θ denote the polar coordinates defined by $x=r\cos\theta$ and $y=r\sin\theta$, the temperature and the internal heat source generation rate are assumed to be independent of θ, given by $T(r,z,t)$ and $Q(r,z,t)$, respectively. The thermal conductivity is functionally graded in the radial and axial directions of the solid of revolution and is taken to be temperature dependent, such that

$$\kappa = g(r,z)h(T), \tag{2}$$

where g is a suitably given function which is positive in R and $h(T)$ is a function which is integrable with respect to T. The density ρ and the specific heat capacity c are also dependent on r, z and T.

Mathematically, the problem of interest here is to solve (1) together with (2) subject to the initial-boundary conditions

$$T(r,z,0)=f_0(r,z) \quad \text{in } R,$$

$$T(r,z,t)=f_1(r,z,t) \quad \text{on } S_1 \text{ for } t>0,$$

$$g(r,z)h(T)\frac{\partial T}{\partial n}=f_2(r,z,t) \quad \text{on } S_2 \text{ for } t>0, \tag{3}$$

where S_1 and S_2 are nonintersecting surfaces such that $S_1 \cup S_2 = S$, S is the (surface) boundary of the region R, $\partial T/\partial n$ denotes the outward normal derivative of T on S and $f_0(r,z)$, $f_1(r,z,t)$ and $f_2(r,z,t)$ are suitably given functions.

3. Transformed equations

Through the use of Kirchhoff's transformation, that is

$$\Theta(r,z,t)=\int h(T)\,dT \equiv \mathcal{K}(T), \tag{4}$$

the nonlinear governing partial differential equation defined by (1) and (2) can be rewritten as

$$g\nabla^2\Theta = -Q-\nabla g\cdot(\nabla\Theta)+S(r,z,\Theta)\frac{\partial\Theta}{\partial t}, \tag{5}$$

where

$$S(r,z,\Theta)=\frac{\rho(r,z,\mathcal{M}(\Theta))c(r,z,\mathcal{M}(\Theta))}{h(\mathcal{M}(\Theta))}, \tag{6}$$

if one assumes that (4) can be inverted to give the temperature as $T=\mathcal{K}^{-1}(\Theta)=\mathcal{M}(\Theta)$.

Furthermore, with the substitution

$$\Theta=\frac{1}{\sqrt{g}}\psi, \tag{7}$$

(5) becomes

$$\nabla^2\psi = -\frac{Q}{\sqrt{g}}+B(r,z)\psi+D(r,z,\psi)\frac{\partial\psi}{\partial t}, \tag{8}$$

where ∇^2 is the Laplacian differential operator and

$$B(r,z)=\frac{1}{\sqrt{g(r,z)}}\nabla^2(\sqrt{g(r,z)}), \quad D(r,z,\psi)=\frac{1}{g}S\left(r,z,\frac{1}{\sqrt{g}}\psi\right). \tag{9}$$

The function g is assumed to be such that $\nabla^2(\sqrt{g})$ exists in the solution domain R.

As ψ is a function of r, z and t, Eq. (8) can be written out more explicitly as

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=-\frac{Q(r,z,t)}{\sqrt{g(r,z)}}+B(r,z)\psi+D(r,z,\psi)\frac{\partial\psi}{\partial t}. \tag{10}$$

For the problem under consideration here, as pointed out earlier in Section 2, the solution domain R and its boundary S can be obtained by rotating, respectively, a two-dimensional region and a curve by an angle of 360° about the z-axis. On the rz plane, the two-dimensional region and the curve are denoted by Ω and Γ, respectively. Fig. 1 gives a sketch of Ω (shaded region) and Γ. In Fig. 1, Γ is an open curve having endpoints A and B on the z-axis. In general, Γ may also be a closed curve, as in, for example, the case in which R is the hollow cylindrical region defined by $u<r<v$, $0<z<w$, where u, v and w are positive constants.

In view of (4) and (7), the initial-boundary conditions in (3) can be rewritten on the rz plane as

$$\psi(r,z,0)=\sqrt{g(r,z)}\mathcal{K}(f_0(r,z)) \quad \text{in } \Omega,$$

$$\psi(r,z,t)=\sqrt{g(r,z)}\mathcal{K}(f_1(r,z,t)) \quad \text{on } \Gamma_1 \text{ for } t>0,$$

$$\frac{\partial}{\partial n}[\psi(r,z,t)]=\frac{\psi(r,z,t)}{2g(r,z)}\frac{\partial}{\partial n}[g(r,z)]+\frac{1}{\sqrt{g(r,z)}}f_2(r,z,t) \quad \text{on } \Gamma_2 \text{ for } t>0, \tag{11}$$

where Γ_1 and Γ_2 denote the curves that can be rotated by an angle of 360° about the z-axis to generate the surfaces S_1 and S_2, respectively, and

$$\frac{\partial}{\partial n}[\psi(r,z,t)]=n_r(r,z)\frac{\partial}{\partial r}[\psi(r,z,t)]+n_z(r,z)\frac{\partial}{\partial z}[\psi(r,z,t)],$$

$$\frac{\partial}{\partial n}[g(r,z)]=n_r(r,z)\frac{\partial}{\partial r}[g(r,z)]+n_z(r,z)\frac{\partial}{\partial z}[g(r,z)], \tag{12}$$

where $n_r(r,z)$ and $n_z(r,z)$ are the components of the outward unit normal vector on Γ at the point (r,z) in the r and z directions, respectively.

Once $\psi(r,z,t)$ (hence $\Theta(r,z,t)$) is obtained by solving (10) in Ω subject to the initial-boundary conditions in (11), the temperature $T(r,z,t)$ may be obtained by inverting Kirchhoff's transformation in (4).

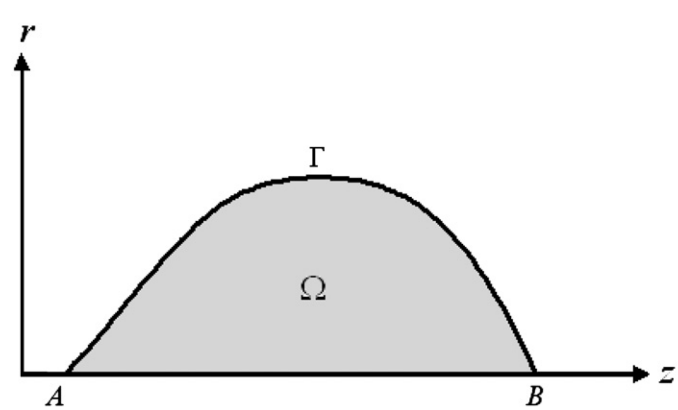

Fig. 1. A sketch of Ω and Γ.