Engineering Analysis with Boundary Elements 34 (2010) 747–754

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A boundary meshless method with shape functions computed from the PDE

Djédjé Sylvain Zézé [a,*], Michel Potier-Ferry [a], Noureddine Damil [b]

[a] Laboratoire de Physique et Mécanique des Matériaux, FRE CNRS 3236, Université Paul Verlaine-Metz, Ile du Saulcy, 57045 Metz Cedex 01, France
[b] Laboratoire de Calcul Scientifique en Mécanique, Faculté des Sciences Ben M'Sik, Université Hassan II Mohammedia - Casablanca, B.P. 7955, Sidi Othman, Casablanca, Morocco

ARTICLE INFO

Article history:
Received 30 July 2009
Accepted 24 March 2010
Available online 21 April 2010

Keywords:
Perturbation technique
Least-squares method
Meshless
High degree polynomials
Boundary-only

ABSTRACT

This paper presents a new meshless method using high degree polynomial shape functions. These shape functions are approximated solutions of the partial differential equation (PDE) and the discretization concerns only the boundary. If the domain is split into several subdomains, one has also to discretize the interfaces. To get a true meshless integration-free method, the boundary and interface conditions are accounted by collocation procedures. It is well known that a pure collocation technique induces numerical instabilities. That is why the collocation will be coupled with the least-squares method. The numerical technique will be applied to various second order PDE's in 2*D* domains. Because there is no integration and the number of shape functions does not increase very much with the degree, high degree polynomials can be considered without a huge computational cost. As for instance the p-version of finite elements or some well established meshless methods, the present method permits to get very accurate solutions.

1. Introduction

This paper presents a discretization technique to solve elliptic partial differential equations. It relies on high degree shape functions that are approximate solutions of the PDE. More precisely one requires that the residual is of order $O(|x-c|^{N+1})$ in the neighborhood of a given center c. This permits to discretize only the boundary conditions. These discretized equations could be obtained by two classical ways, first Galerkin procedure with the drawback of the integration cost, second point-collocation that is integration-less, but can lead to numerical instabilities and ill-conditioned matrices [7]. In this paper point-collocation method has been chosen and associated with a least-squares minimization to overcome the numerical instabilities, as proposed by Zhang et al. [18] and used by several others [19–24]. So the proposed method is characterised by closed-form solutions built by Taylor series, boundary discretization and coupling between collocation and least-squares minimization.

To our best knowledge, such a discretization principle has never been presented in the numerical literature, but of course it has some points in common with the many numerical methods that are not based on low degree polynomials. It can be compared with the p-version of finite element and one can hope that the presented technique permits to recover more or less the same accuracy and adaptivity as the p-version [1–3]. The differences lie in the number of shape functions that is much smaller with the present method, in the computation cost and in the discretization principle. Our discrete problem is deduced from point-wise equations as in many meshless methods [4–6,8,25], but the present method does not use a priori given shape functions, they are built from a local solving of the PDE. There are at least three well known numerical methods that associate a family of exact solutions and a boundary discretization: the integral equation method [9–11], the method of fundamental solutions [12–14] and the scaled boundary finite element method [15–17]. In these three methods, the reference problem has to be linear with constant coefficients while the present Taylor series method can be extended to generic PDE's.

The paper is organised as follows. In the second part, the instabilities due to pure collocation are pointed out and compared with least-squares collocation. In Part 3, a computational technique is sketched that permits to apply Taylor series to PDE's. Finally in Part 4, various 2*D* applications are discussed to assess the possibilities of the presented numerical method.

2. Boundary collocation versus boundary least-squares collocation

2.1. Polynomial shape functions

Let us consider the Dirichlet problem in a 2*D* domain:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(\underline{x}) = u^d(\underline{x}) & \text{on } \partial\Omega \end{cases} \tag{1}$$

* Corresponding author.
E-mail address: s.zeze@xerya.fr (D.S. Zézé).

0955-7997/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2010.03.008

The aim is to introduce high degree polynomial shape functions, that are exact solutions of the considered partial differential equations. In the case of the Laplace equation $\Delta u=0$, there are only two such polynoms of degree n, namely $Re(x+iy)^n$ and $Im(x+iy)^n$. Next, let us introduce all the polynoms, whose degree is lower or equal to p. The dimension of this vectorial space is $((p+1)(p+2))/2$, but if one limits to the solutions of the Laplace equation, this dimension is reduced to $2p+1$. Note that the limitation to the solutions of the PDE permits to reduce strongly the number of shape functions, for instance 101 polynoms instead of 1326 for a degree equal to 50. With this reduction of the number of functions, one can hope to build a numerical method that remains efficient with a large degree.

2.2. Boundary collocation

As for instance in the method of fundamental solutions (MFS), exact solutions of the PDE are used. Hence it is only necessary to discretize the boundary conditions. Hence, the cloud of collocation points is located on the boundary. The simplest technique is to choose as many collocation points $\underline{x}_i$ as shape functions $P_i(\underline{x}_i)$. The unknown is written in the classical form as

$$u(\underline{x}) = \sum_{i=1}^{2p+1} P_i(\underline{x}) v_i \tag{2}$$

and the discretized equations are

$$\sum_{i=1}^{2p+1} P_i(\underline{x}_j)\underline{v}_i = u^d(\underline{x}_j), \quad 1 \le j \le 2p+1 \tag{3}$$

Let us apply, this simple boundary collocation to a unit disk $x^2+y^2 \le 1$, and with the boundary data

$$u^d(x,y) = \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} \tag{4}$$

The exact solution is known: $u_{ex}(x,y)=(x-x_0)/((x-x_0)^2+(y-y_0)^2)$.

First, one chooses a uniformly distributed cloud (see Fig. 2). In Fig. 1, we have plotted the error $|(u(x,0)-u_{ex}(x,0))/u_{ex}(x,0)|$ along the horizontal axis for three values of the degree. In this case, the boundary collocation method converges with the order p, see Fig. 1. For instance, for $p=32$, the maximal error is about 10^{-3} and the error in the center of the disk is about 10^{-6}. The same conclusion holds also by looking at the error anywhere in the domain. Unfortunately, this simple collocation technique is not robust and it does not work with an irregular cloud. For instance, for $p=10$ and the collocation points of Fig. 3, the maximal value of the approximated solutions by this boundary collocation technique is about 1075, instead of 4 for the exact one. It is not surprising that this simple collocation technique does not work. Indeed in the present example of a disk, the boundary value of the polynom is given by a truncated Fourier series and the coefficients v_i are identical to the Fourier coefficients. In the present technique, one tries to identify the Fourier coefficients from pointwise data and with about two points per period $2\pi/p$, which is not sufficient for a stable estimate. Theoretically, the Fourier coefficients are given by integral formulae:

$$\begin{cases} \dfrac{1}{\pi}\int_0^{2\pi} u(\theta)\cos n\theta \, d\theta \\ \dfrac{1}{\pi}\int_0^{2\pi} u(\theta)\sin n\theta \, d\theta \end{cases} \tag{5}$$

To avoid the numerical evaluation of these integrals (5) that involves many integration points, we shall propose to identify these Fourier series from a number of pointwise data that is larger than $2p+1$ (Fig. 2).

2.3. Boundary least-squares collocation

It is proposed to identify the coefficients v_i of the polynom (2) from M collocation points, M being larger than $2p+1$. The Dirichlet boundary condition will be satisfied in a least-square sense. Such a least-square collocation method has been presented by Zhang et al. [18] in another meshless framework and it has been widely applied. One requires that the coefficients v_i minimize the function

$$J(v_i) = \frac{1}{2}\sum_{j=1}^{M} |u(\underline{x}_j)-u^d(\underline{x}_j)|^2 = \frac{1}{2}\sum_{j=1}^{M}\left|\sum_{i=1}^{2p+1} P_i(\underline{x}_j)v_i - u^d(\underline{x}_j)\right|^2 \tag{6}$$

After few calculations, this minimization of (6) leads to a linear system

$$[K]\{v\}=\{b\} \tag{7}$$

where

$$[K] = \sum_{j=1}^{M}[K^j] \quad \text{with } K^j_{ik} = P_i(\underline{x}_j)P_k(\underline{x}_j) \tag{8a}$$

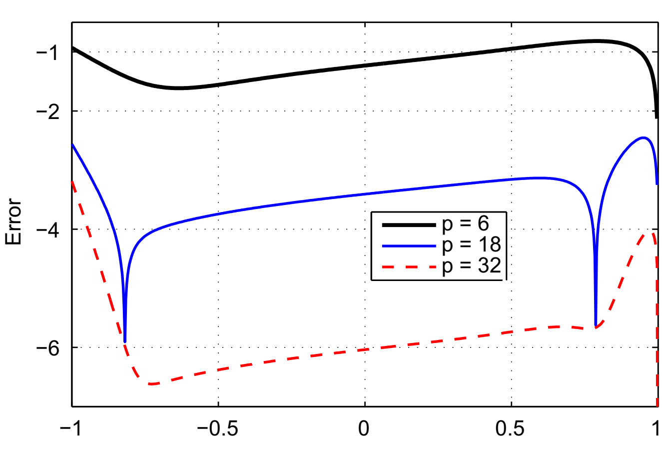

Fig. 1. Dirichlet problem in a disk. Pure boundary collocation with a uniform cloud, see Fig. 2. Error along the horizontal axis.

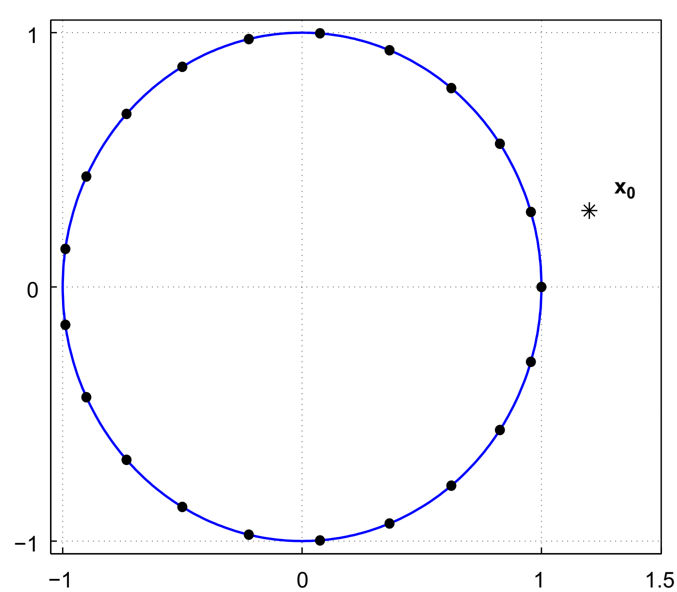

Fig. 2. Uniform distribution of collocation points, $x_0=1.2$, $y_0=0.3$.