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Computation of singular and hypersingular boundary integrals by Green identity and application to boundary value problems

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ABSTRACT

The problem of computing singular and hypersingular integrals involved in a large class of boundary value problems is considered. The method is based on Green's theorem for calculating the diagonal elements of the resulting discretized matrix using the Nyström discretization method. The method is successfully applied to classical boundary value problems. Convergence of the method is also discussed.

1. Introduction

The mathematical treatment of the scattering of time-harmonic acoustic or electromagnetic waves by an infinitely long cylindrical obstacle with a simply connected bounded cross-section $\Omega \subset \mathbf{R}^2$, leads to the exterior Helmholtz equation

$$(\Delta + k^2)u = 0 \quad \text{in } \mathbf{R}^2 \backslash \Omega, \tag{1.1}$$

where $k>0$ is the wave number and the field u is decomposed, $u = u^{inc} + u^s$, into the given incident field u^{inc}, which is assumed to satisfy the Helmholtz equation everywhere, except possibly at isolated points in $\mathbf{R}^2$, and the unknown scattered field u^s, which is required to satisfy the Sommerfeld radiation condition

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{1/2}\left(\frac{\partial u^s}{\partial |\mathbf{x}|} - iku^s\right) = 0, \tag{1.2}$$

uniformly in all directions. Depending on the physical nature of the scattering obstacle, the total field u has to satisfy a boundary condition on Γ, the boundary of Ω. Usually the following boundary conditions are considered

1. Dirichlet condition:

$$u = 0; \tag{1.3}$$

2. Neumann condition:

$$\frac{\partial}{\partial v}u = 0, \tag{1.4}$$

where v is the unit outward normal, which is assumed to be directed to the exterior; and

3. Transmission condition:

$$\frac{\partial}{\partial v}u = \rho\frac{\partial}{\partial v}u_i, \tag{1.5}$$

$$u = u_i, \tag{1.6}$$

where ρ is a non-zero complex constant and u_i satisfies the Helmholtz equation in Ω, with k replaced by k_i.

In acoustics the Dirichlet (resp. Neumann) condition corresponds to sound soft (resp. hard), whereas in electromagnetics it models scattering from a perfect conductor with the electromagnetic field H (resp. E)-polarized. The transmission condition corresponds in acoustics (resp. electromagnetics) to the continuity of pressure and normal velocity (resp. scattering from a dielectric).

The problem of determining the unknown function u^s, for the above stated equations, with arbitrary shape of Ω and wave number k, is not an easy task. Usually a tedious and costly numerical calculation is expected.

Many numerical solutions have been considered. The finite element and finite difference methods could be used to compute the unknown field, but they involve a discretization of the two-dimensional (2D) space, which is a heavy numerical task, especially for high wave numbers. Moreover it is impossible to discretize to infinity, and one has either to couple the finite element with boundary element using artificial boundaries or select a cut-off at some arbitrary distance from the obstacle and implement there the Sommerfeld radiation condition (cf. [15] and the references therein).

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One of the most popular strategies in recent years is the plane-wave decomposition method (PWDM) [12,26]. The method is based on the assumption that each (discretized) solution can be approximated by an ansatz of the superposition of plane waves, at a given wave number. The ansatz solves the Helmholtz equation in the discretized system and the boundary condition determines the unknown functions. This method, thus, reduces the calculations to a one-dimensional (1D) boundary grid instead of the 2D space. The problem of creating discretizations ("meshing") is well known to be a difficult task—almost an art, and the simplicity achieved by a reduction in dimensionality must not be underestimated. The PWDM is found to be extremely efficient in practice. However, solutions of non-convex boundaries, in general, cannot be approximated by any solution of the Helmholtz equation regular everywhere (in particular, by linear combinations of a finite number of plane waves having the same energy) [11]. From this we see that PWDM cannot be applied efficiently to important types of shapes.

The main useful numerical strategies that have been suggested in the literature, for arbitrary shapes, are based on boundary integral approach and often referred to as boundary integral method (BIM). This strategy, which requires 1D grid calculations, is very similar to the PWDM. It is also based on the observation that the solutions are completely determined by their behavior at the boundary, and use basis functions that satisfy the Helmholtz equation in the system at fixed wave number. A linear combination of the basis functions is then selected such that the boundary conditions are satisfied. The method is derived from exact integral equation using Green's theorem and/or layer potentials [4]. Several versions of the BIM exist depending on the choice of basis functions and discretization [17,6,13].

A hybrid boundary method unifying BIM and PWDM has been developed in [6]. This method, called the gauge freedom method (GFM), has as its main advantage the choice of a regular kernel in the integral equation.

BIM approaches are fundamental tools in both the numerical solution and in the theoretical analysis. Uniqueness and existence theorems are often much easier established in this way.

Another major advantage of the integral equation representation for external problems is that these ensure that the Sommerfeld radiation condition is automatically satisfied exactly. While considerable progress has been made in developing the so-called perfectly matched layers to imitate the properties of the far-field, these are relatively difficult to implement.

Despite these advantages the integral equation approaches have some minor disadvantages. The first is that their formulation is relatively more complex mathematically. However, this need not be an obstacle to their understanding and implementation, since there are many relatively clear expositions of the integral equation approaches. A second disadvantage of the integral equation approach is that it leads to linear systems with dense matrices. For large sizes, these dense matrices are relatively expensive to perform computations with. Many modern calculations require characterization of the scattering off complex shaped objects, where large matrices arise. The fast multipole methods (FMM) [23] allow computation of the product of a vector with a dense matrix of the kind that arises upon discretization of the integral equation to be done extremely rapidly, and go a long way towards alleviating this disadvantage.

The most crucial issue and major problem in BIM implementations is the evaluation of singular and hypersingular integrals involved in the integral equations. These integrals contribute to the dominant (diagonal and near diagonal) terms of the discretized boundary element matrices. The treatment of singular and hypersingular integrals has been a subject of investigation for the development of BIM in the past decades, and many techniques have been proposed so far. We will make no attempt to review these many contributions (see, e.g., [22] and the references therein), as this goes beyond the scope of this paper. These methods include analytical integration if possible [22,18], approximation by equations with smooth kernels [7,25] and subtraction of the singularity [2].

In this paper we consider a new approach where we can compute the singular and hypersingular parts of the integrals involved in BIM by a simple use of Green's theorem and particular solutions of the Helmholtz equations. We show computational results for 2D domains, but the method can be easily extended to the three-dimensional (3D) case. The numerical results will be compared to the very efficient method in [18,5, Chapter 3.5], where analytical computation of the diagonal elements with a Nyström discretization were performed. We will demonstrate that our method has exponential convergence, and is much easier to implement, especially for the hypersingular kernel, compared to the previously mentioned methods.

The paper is organized as follows. In Section 2 we recall the boundary integrals and discuss briefly the derivation of boundary integral equations. In Section 3 we implement the numerical algorithm and discuss few examples. Finally, in Section 4, we summarize the results.

2. Boundary integral formulation

First, let us introduce the boundary integral operators. We denote the fundamental solution of Eq. (1.1) (the free-space source) by

$$\Phi(\mathbf{x},\mathbf{y}) = -\frac{i}{2}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|),$$

where $H_0^{(1)}$ is the Hankel function of order zero and the first kind.

For $\phi,\psi \in C(\Gamma)$, define the single and double layer potentials

$$S\phi(\mathbf{x}) = \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}), \quad \mathbf{x}\in\mathbf{R}^2\backslash\Gamma \tag{2.1}$$

and

$$D\psi(\mathbf{x}) = \int_\Gamma \frac{\partial}{\partial\nu(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,ds(\mathbf{y}), \quad \mathbf{x}\in\mathbf{R}^2\backslash\Gamma \tag{2.2}$$

with densities ϕ and ψ, respectively. Their normal derivatives are given by

$$M\phi(\mathbf{x}) = \frac{\partial}{\partial\nu(\mathbf{x})}S\phi(\mathbf{x}) \quad \text{and} \quad N\psi(\mathbf{x}) = \frac{\partial}{\partial\nu(\mathbf{x})}D\psi(\mathbf{x}). \tag{2.3}$$

It is known (cf. [4, Sections 2.4 and 2.5]) that the above defined potentials are analytic in $\mathbf{R}^2\backslash\Gamma$ and when $\mathbf{x}$ approaches Γ, S and N are continuous, whereas D and M exhibit jumps. In particular, on Γ,

$$S\to\hat{S}, \quad N\to\hat{N}, \quad D\to\mp I+\hat{D} \quad \text{and} \quad M\to\pm I+\hat{M}, \tag{2.4}$$

where the upper (resp. lower) sign corresponds to the limit when $\mathbf{x}$ approaches Γ from outside (resp. inside) and I is the identity operator. The compact operators $\hat{S},\hat{M},\hat{D}: C(\Gamma)\to C(\Gamma)$ are given, on Γ, by

$$\hat{S}\phi(\mathbf{x}) = \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}), \quad \mathbf{x}\in\Gamma,$$

$$\hat{M}\phi(\mathbf{x}) = \int_\Gamma \frac{\partial}{\partial\nu(\mathbf{x})}\Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}), \quad \mathbf{x}\in\Gamma,$$

$$\hat{D}\psi(\mathbf{x}) = \int_\Gamma \frac{\partial}{\partial\nu(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,ds(\mathbf{y}), \quad \mathbf{x}\in\Gamma$$