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Models of corner and crack singularity of linear elastostatics and their numerical solutions

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ABSTRACT

The singular solutions for linear elastostatics at corners are *essential* in both theory and computation. In this paper we seek the singular solutions for corners with the clamped and the free stress boundary conditions, and explore corner singularity in detail. In this paper the singular solutions of linear elastostatics are derived, and two new models of interior crack singularity are proposed. The collocation Trefftz methods are used to obtain highly accurate solutions, where the leading coefficient has 14 (or 12) significant digits by the computation with double precision. Such solutions are useful to examine other numerical methods for singularity problems in linear elastostatics. Also the explicit singular solutions can be adapted to design and develop efficient numerical methods for singularity problems, such as the combined method (Li, 1998, 2008 [19,20]) and the Trefftz methods which include the boundary approximation method (Li, 1990, Li et al., 1987 [18,26]), the collocation Trefftz method (Li et al., 2008 [24]), the hybrid Trefftz method (Qin, 2000 [36]), the boundary collocation techniques (Kolodziej and Zielinski, 2009 [16]), etc. This paper also explores a systematic analysis for singularity properties and explicit singular solutions for corners of linear elastostatics.

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1. Introduction

The traditional finite element method (FEM) and finite difference method (FDM) provide poor accuracy of numerical solutions for singularity problems. Many innovative methods have been developed, to seek the numerical solutions with optimal convergence rates and good stability. For Poisson's equation and other elliptic equations, a systematic study is provided in Li [19]. The singular properties and the singular solutions near the corners are crucial to design effective numerical methods. Moreover, highly accurate solutions are important to examine different numerical methods. For Laplace's equation, Motz's problem is a benchmark of singularity problems, and its highly accurate solutions are provided in [19,23,26,30] by the collocation Trefftz method (CTM).

For biharmonic equation, similar models to Motz's problem are first developed in [23], and stability analysis is explored in [25]. The singular solutions near corners under free stress boundary conditions were first given in Williams [38], and then in Lin and Tong [28], Jirousek and Venkstesh [14], Jirousek and Wroblewski [15], Piltner [32], Drombosky et al. [8], and Qin [36]. In this paper, our efforts are paid to linear elastostatics, and to derive new particular solutions of linear elastostatics near corners under the clamped, and the free stress boundary conditions, in addition two new crack models are proposed. It is worthy pointing out that the singular solutions near corners in this paper directly from free stress boundary conditions are coincident with those in [38,28] from biharmonic equations by using a similarity mapping.

The singular properties and the explicit singular solutions of linear elastostatics at corners are *essential* in both theory and computation. Once the singularity of corner solutions is known, the reduced convergence rates of FEM, FDM, FVM and other numerical methods are found (see Section 7.2), and some improved techniques can be invented, to recover the optimal convergence rates (see [19]). Moreover, once the explicit singular solutions are known, some models as in Sections 4 and 6 can be designed, and the collocation Trefftz method can be used to give their highly accurate solutions, which can be used for testing other numerical methods (also see [19]). More importantly, based on the explicit particular solutions of corners given in this paper, we may develop a number of efficient numerical methods for linear elastostatics, such as the Trefftz methods including the collocation Trefftz method and the hybrid Trefftz method, and the combinations (see [1,4,7–9,12,17,20,27,29,34,35,39].)

This paper is organized as follows. In Section 2, a basic description for elastostatics problems in 2D is introduced, and their particular solutions are provided. In Section 3, singular

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solutions near corners are derived for the clamped boundary conditions. In Section 4, a model of crack singularity with clamped boundary conditions is designed, and its numerical solutions are sought by the collocation Trefftz method (CTM) as in [19,24]. In Section 5, singular solutions near corners are derived for the free stress boundary conditions, and in Section 6 the other model of crack singularity with free stress is designed, and numerical results are also provided. In Section 7, the algorithms are given for the leading powers ν_k of the corner solution $O(r^{\nu_k})$, and numerical results are provided for the rectangular corner and the concave corner of the L-shaped domain. In the last section, a few concluding remarks are made.

2. Linear elastostatics problems in 2D

2.1. Basic equations

Consider the linear elastostatics problem in 2D. Denote the displacement vector,

$$\vec{w} = \mathbf{w} = \{w_1(\mathbf{x}), w_2(\mathbf{x})\}^T = \{u(x,y), v(x,y)\}^T, \tag{2.1}$$

where $\vec{x} = \mathbf{x} = (x_1, x_2) = (x, y)$. The linear strain tensor is given by

$$\varepsilon_{ij}(\mathbf{x}) = \frac{1}{2}\left[\frac{\partial w_i(\mathbf{x})}{\partial x_j} + \frac{\partial w_j(\mathbf{x})}{\partial x_i}\right], \quad 1 \le i,j \le 2. \tag{2.2}$$

Let σ_{ij} $(1 \le i,j \le 2)$ denote the stress tensor at $\mathbf{x}$. For an isotropic homogeneous Hookean solid, there exist the stress–strain relations

$$\sigma_{ij} = \lambda(\nabla \cdot \vec{w})\delta_{ij} + 2\mu\varepsilon_{ij}, \quad 1 \le i,j \le 2, \tag{2.3}$$

where "$\nabla \cdot$" is the divergence operator, λ and μ are the Lamé constants, and δ_{ij} is the Kronecker delta.

When there exists a body force $\vec{f}$, we obtain the nonhomogeneous equation, called the Lamé system for isotropic body:

$$\mu\Delta\vec{w} + (\lambda+\mu)\nabla(\nabla \cdot \vec{w}) + \vec{f} = 0 \quad \text{in } S. \tag{2.4}$$

When $\vec{f} \equiv \vec{0}$, we have the Cauchy–Navier equation of linear elastostatics:

$$\Delta\vec{w} + \frac{1}{1-2\nu}\nabla(\nabla \cdot \vec{w}) = 0 \quad \text{in } S, \tag{2.5}$$

where the Poisson ratio

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad 0 < \nu < \frac{1}{2}. \tag{2.6}$$

Young's modulus E and the bulk modulus K are introduced by

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad K = \frac{E}{3(1-2\nu)}. \tag{2.7}$$

The inverse relations of (2.6) and (2.7) are given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{2.8}$$

The strain–stress relations are given by

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sum_{k=1}^{2}\sigma_{kk}. \tag{2.9}$$

There also exist the symmetric relations:

$$\sigma_{ij} = \sigma_{ji}, \quad \varepsilon_{ij} = \varepsilon_{ji}. \tag{2.10}$$

Denote the constant

$$\kappa = \frac{1}{4(1-\nu)}. \tag{2.11}$$

For the plane strain problem the constant is given by

$$D = \frac{\lambda+\mu}{\lambda+3\mu} = \frac{1}{3-4\nu} = \frac{\kappa}{1-\kappa}, \tag{2.12}$$

and for the plane stress problem,

$$D = \frac{1}{3-4\nu} = \frac{1+\hat{\nu}}{3-\hat{\nu}}, \quad \nu = \frac{\hat{\nu}}{1+\hat{\nu}}. \tag{2.13}$$

.

2.2. Traction boundary conditions

The Cauchy–Navier equation (2.5) is written explicitly by

$$\mu\Delta u + (\lambda+\mu)\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right\} = 0 \quad \text{in } S, \tag{2.14}$$

$$\mu\Delta v + (\lambda+\mu)\left\{\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right\} = 0 \quad \text{in } S, \tag{2.15}$$

or by

$$\Delta u + \frac{1}{1-2\nu}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right\} = 0 \quad \text{in } S, \tag{2.16}$$

$$\Delta v + \frac{1}{1-2\nu}\left\{\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right\} = 0 \quad \text{in } S, \tag{2.17}$$

where ν is given in (2.6). The traction on ∂S is given by

$$\vec{\tau}(\vec{w})(\mathbf{x}) = (\tau_1(u,v), \tau_2(u,v))^T, \tag{2.18}$$

where the components

$$\tau_1(u,v) = \sigma_x n_1 + \sigma_{xy} n_2 = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)n_1 + 2\mu\frac{\partial u}{\partial \nu} + \mu n_2\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{2.19}$$

$$\tau_2(u,v) = \sigma_{xy} n_1 + \sigma_y n_2 = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)n_2 + 2\mu\frac{\partial v}{\partial \nu} - \mu n_1\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{2.20}$$

where $n_1 = \cos(\nu, x)$, $n_2 = \cos(\nu, y)$, and the stress

$$\sigma_x = \sigma_{11}, \quad \sigma_y = \sigma_{22}, \quad \sigma_{xy} = \sigma_{12} = \sigma_{21}. \tag{2.21}$$

2.3. Particular solutions

In Jirousek and Wroblewski [15], Jirousek and Venkstesh [14] and Qin [36], for the plane stress equations (2.16) and (2.17), the particular solutions are expressed as the complex functions. Denote $i = \sqrt{-1}, \mathbf{z} = x + iy = re^{i\theta}, \overline{\mathbf{z}} = x - iy = re^{-i\theta}$. The particular solutions $u(x,y)$ and $v(x,y)$ of the plane stress equations are given by the real and imaginary parts of A_k, B_k, C_k and D_k below, respectively,

$$A_k = i\mathbf{z}^k + iDk\mathbf{z}\overline{\mathbf{z}}^{k-1}, \tag{2.22}$$

$$B_k = \mathbf{z}^k - Dk\mathbf{z}\overline{\mathbf{z}}^{k-1}, \tag{2.23}$$

$$C_k = i\overline{\mathbf{z}}^k, \tag{2.24}$$

$$D_k = -\overline{\mathbf{z}}^k, \quad k = 1, 2\ldots, \tag{2.25}$$

where D is given in (2.12) or (2.13). We have the following linear combination for the Trefftz method (TM):

$$u_L = \sum_{k=1}^{L}\{a_k\Re(A_k) + b_k\Re(B_k) + c_k\Re(C_k) + d_k\Re(D_k)\} + d_0, \tag{2.26}$$

$$v_L = \sum_{k=1}^{L}\{a_k\Im(A_k) + b_k\Im(B_k) + c_k\Im(C_k) + d_k\Im(D_k)\} + c_0, \tag{2.27}$$

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