Engineering Analysis with Boundary Elements 32 (2008) 692–696

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

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An adaptive element subdivision technique for evaluation of various 2D singular boundary integrals

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Received 23 October 2007; accepted 17 December 2007
Available online 17 March 2008

Abstract

In this paper, a unified algorithm is presented for the numerical evaluation of weakly, strongly and hyper singular boundary integrals with or without a logarithmic term, which often appear in two-dimensional boundary element analysis equations. In this algorithm, the singular boundary element is broken up into a few sub-elements. The sub-elements involving the singular point are evaluated analytically to remove the singularities by expressing the non-singular parts of the integration kernels as polynomials of the distance r, while other sub-elements are evaluated numerically by the standard Gaussian quadrature. The number of sub-elements and their sizes are determined according to the singularity order and the position of the singular point. Numerical examples are provided to demonstrate the correctness and efficiency of the proposed algorithm.

Keywords: Singular integrals; Boundary element; Subdivision technique; Gaussian quadrature

1. Introduction

When the boundary integral equation method is used to solve potential and mechanical problems, weakly and strongly singular integrals are included in the basic boundary integral equations [1–5], and hypersingular integrals are involved in the gradient integral equations [6–10]. Accurately evaluating these singular integrals is a crucial task in the boundary element method (BEM). Much effort has been made to remove the singularities appearing in these singular integrals [11–20]. The common idea behind these methods is based on the singularity separation scheme by adding and subtracting singular kernel functions. The separated singular integrals are regularized by the local coordinate technique [9–12], global coordinate technique [3,7,8,13], and distance transformation method [14,15]. Among these methods, the local regularization technique initiated by Guiggiani et al. [9,10] is extensively used due to its flexibility for handling various orders of singularities. The main drawback of this technique is that it requires the expansion of every quantity involved in the integrand as Taylor's series about the local distance ρ. The maximum number of the expanded Taylor's series is determined by the order of the shape functions. So far, terms up to ρ^2 are used based on commonly used quadratic elements. If more expansion terms are desired, higher order shape functions should be employed and consequently the Taylor's expansion task may become prohibitively complicated.

Recently, Gao [20] presented a simple technique for evaluating various two-dimensional (2D) singular boundary integrals by expressing the non-singular part of the integrand as polynomials of the global distance r and removing various singularities in a unified way through analytically integrating the resulting integrals about r. This technique is simple and powerful, but it requires more polynomial terms in r when handling complicated integrands and highly curved elements. This may give rise to difficulties in choosing an adequate number of polynomials.

In this paper, an adaptive algorithm is presented by dividing the boundary element over which the source point is located into a number of sub-elements. Singularities in

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0955-7997/$ - see front matter © 2008 Published by Elsevier Ltd.
doi:10.1016/j.enganabound.2007.12.004

the singular sub-element which includes the source point are removed analytically by expressing the non-singular part of the integrand as polynomials in r as done in Ref. [20], while the remaining regular sub-elements are evaluated using the standard Gaussian quadrature. Based on a criterion stemming from error analysis and numerical investigation into the Gaussian quadrature, the element subdivision procedure on the number of sub-elements and their sizes is automatically carried out. Since the subdivision technique is used and consequently the singular sub-element is relatively small, the expansion of the integrand into power series requires much fewer terms than the technique based on the whole element expansion in Ref. [20]. Computation shows that even the quadratic expansion can give satisfactory results for the weakly, strongly and hyper singular boundary integrals. A number of numerical examples are given to verify the validity of the presented algorithm.

2. Various 2D singular boundary integrals

In the BEM analysis, usually the following type of boundary integrals is encountered [1–5]:

$$K_i(\boldsymbol{x}^p) = \int_\Gamma f_i(\boldsymbol{x}^p, \boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) = \int_\Gamma \bar{f}_i(\boldsymbol{x}^p, \boldsymbol{x}) \log[r(\boldsymbol{x}^p, \boldsymbol{x})]\,\mathrm{d}\Gamma(\boldsymbol{x}) \tag{1a}$$

$$I_i(\boldsymbol{x}^p) = \int_\Gamma f_i(\boldsymbol{x}^p, \boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) = \int_\Gamma \frac{\bar{f}_i(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})}\,\mathrm{d}\Gamma(\boldsymbol{x}) \tag{1b}$$

$$J_i(\boldsymbol{x}^p) = \int_\Gamma f_i(\boldsymbol{x}^p, \boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) = \int_\Gamma \frac{\bar{f}_i(\boldsymbol{x}^p, \boldsymbol{x}) \log[r(\boldsymbol{x}^p, \boldsymbol{x})]}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})}\,\mathrm{d}\Gamma(\boldsymbol{x}), \tag{1c}$$

where Γ is the boundary (a straight or curved line) of the problem to be considered, $\boldsymbol{x}^p$ denotes the source point and $\boldsymbol{x}$ the field point. The function $\bar{f}_i(\boldsymbol{x}^p, \boldsymbol{x})$ is assumed to be bounded everywhere and $r(\boldsymbol{x}^p, \boldsymbol{x})$ is the distance between points $\boldsymbol{x}^p$ and $\boldsymbol{x}$, i.e.,

$$r(\boldsymbol{x}^p, \boldsymbol{x}) = ||\boldsymbol{x} - \boldsymbol{x}^p|| = \sqrt{(x - x^p)^2 + (y - y^p)^2}. \tag{2}$$

The boundary integral (1a) is weakly singular when the field point $\boldsymbol{x}$ approaches the source point $\boldsymbol{x}^p$, (1b) is strongly singular when $\beta = 1$ and hypersingular when $\beta = 2$, and (1c) is hypersingular when $\beta \geqslant 1$.

The weakly singular boundary integral (1a) always exists (i.e. is finite). However, the strongly and hypersingular boundary integrals only exist under some conditions, depending heavily on the characteristics of the function $\bar{f}_i(\boldsymbol{x}^p, \boldsymbol{x})$ [20]. Usually, integrals stemming from physical problems should always exist. In this paper, strongly singular boundary integrals are interpreted in the Cauchy principal value sense, and hypersingular ones in the Hadamard finite part sense.

3. Evaluation of singular boundary integrals using element subdivision technique

To numerically evaluate boundary integrals, the boundary Γ is discretized into a number of linear or quadratic elements [1–5]. After discretizing the boundary into N_{elem} elements, the boundary integrals (1) can be written as

$$\int_\Gamma f_i(\boldsymbol{x}^p, \boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) = \sum_{e=1}^{N_{\text{elem}}} \int_{\Gamma_e} f_i(\boldsymbol{x}^p, \boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}). \tag{3}$$

The standard Gaussian quadrature is used to evaluate the regular elements which do not include the singular point $\boldsymbol{x}^p$. The element over which the singular point $\boldsymbol{x}^p$ is located is discretized into a number of sub-elements around $\boldsymbol{x}^p$. Fig. 1 shows a case in which four sub-elements are introduced. The sub-element s including point $\boldsymbol{x}^p$ is singular and special treatment is required, while sub-elements e_i are regular and therefore the standard Gaussian quadrature can be used to evaluate them. When the singular point $\boldsymbol{x}^p$ is located within the element as in the cases of the mid-node of quadratic boundary elements and the discontinuous elements, the left part and right part of the element around $\boldsymbol{x}^p$ will be treated separately, each part having the similar pattern as shown in Fig. 1.

3.1. Analytical integration over singular sub-elements

For the sub-element s shown in Fig. 1, the singularities appearing in Eq. (1) can be removed analytically by expressing the non-singular parts of the integration kernels as polynomials of the distance r. To do this, the differentiation element $\mathrm{d}\Gamma$ is expressed in terms of $\mathrm{d}r$ using [20]

$$\mathrm{d}\Gamma = \frac{\mathrm{d}r}{\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{l}}}. \tag{4}$$

where $\hat{\boldsymbol{r}}$ is the unit vector of $\boldsymbol{r}$, and $\hat{\boldsymbol{l}}$ is the unit vector along the tangential direction to $\mathrm{d}\Gamma$ (Fig. 2). In Fig. 2, r_s is the distance between the two ends of the sub-element s.

Let's consider the integral (1b). Other integrals (1a) and (1c) can be treated in a similar manner. Substituting Eq. (4) into Eq. (1b) yields

$$I_i^\varepsilon(\boldsymbol{x}^p) = \int_0^{r_\varepsilon} \frac{\bar{f}_i(\boldsymbol{x}^p, \boldsymbol{x})}{\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{l}} r^\beta}\,\mathrm{d}r. \tag{5}$$

In order to evaluate the singular integrals in the above equation, the non-singular parts of the integrand are

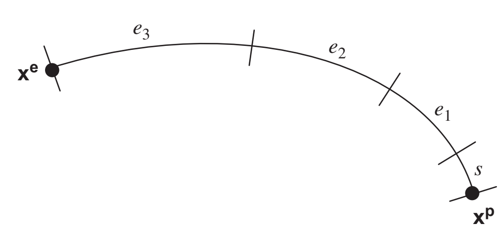

Fig. 1. The singular element is divided into a few sub-elements.