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An integrated-RBF technique based on Galerkin formulation for elliptic differential equations

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ABSTRACT

This paper presents a new radial-basis-function (RBF) technique for solving elliptic differential equations (DEs). The RBF solutions are sought to satisfy (a) the boundary conditions in a local sense using the point-collocation formulation, (b) the governing equation in a global sense using the Galerkin formulation. In contrast to Galerkin finite-element techniques, the present Neumann boundary conditions are imposed in an exact manner. Unlike conventional RBF techniques, the present RBF approximations are constructed "locally" on grid lines through integration and they are expressed in terms of nodal variable values. The proposed technique can provide an approximate solution that is a C^p function across the subdomain interfaces (p—the order of the DE). Several numerical examples are presented to demonstrate the attractiveness of the present implementation.

1. Introduction

The mathematical modelling of engineering problems usually leads to sets of ordinary/partial differential equations (ODEs/PDEs) and their boundary conditions. To seek solutions to differential problems, for most cases, it is necessary to employ discretisation methods to reduce the sets of DEs to systems of algebraic equations. Principal discretisation methods (e.g. finite-difference, finite-element and boundary-element techniques) can be viewed as variants of the method of weighted residuals that can be stated in three well-known formulations, namely the strong, weak and inverse statements [1]. By means of weighting functions in a statement, the residuals for the DE and boundary conditions are made small in some senses. Two popular ways used are (i) the point-collocation approach, where the residuals are zero at certain points, (ii) the Galerkin-type approach, in which the residuals are zero in an average sense over the space of interest. Each approach has some advantages in certain areas of application. The former is cost-effective as no integrations are required, while the latter has a smoothing capability owing to its integral nature.

Radial-basis-function (RBF) collocation methods are considered as a powerful tool for the approximation of scattered data as well as for the solution of differential problems [2]. RBF collocation methods are capable of approximating arbitrarily-well continuous functions. A number of RBFs such as the multiquadric and Gaussian basis functions have spectral approximation power. However, the condition number of the RBF interpolation matrix also grows rapidly with respect to (a) the decrease in distance between the RBF centres, (b) the increase in the RBF width. The methods thus, in practice, suffer from a trade-off between accuracy and stability [3]. Moreover, there is a gap in accuracy between the RBF solutions to Neumann- and Dirichlet-type boundary-value problems. To improve the numerical stability of a RBF solution, there are a number of schemes proposed in the literature: for example, (a) preconditioners (e.g. [4]); (b) local RBF approximations (e.g. [5,6]); (c) compactly-supported RBFs (e.g. [7]); (d) domain decompositions (e.g. [8,9]). Recently, an approximation scheme, which is based on point collocation, Cartesian grids and one-dimensional integrated RBF networks (1D-IRBFNs), has been proposed in [10,11]. A problem domain, which can be regular or irregular, is discretised by a Cartesian grid. Along grid lines, 1D-IRBFNs are constructed to satisfy the governing DE together with boundary conditions in an exact manner. The "local" 1D-IRBFN approximations at a grid node involve only nodal points that lie on the grid lines intersected at that point rather than the whole set of nodes. This scheme allows a larger number of nodes to be employed.

There are very few papers on the use of RBFs in the context of Galerkin approximation [2]. Galerkin RBF techniques have been considered in [2,12–14]. In those works, conventional RBF approximations were employed. A function is decomposed into RBFs; its derivatives are then obtained through differentiation. In this study, we present a new numerical scheme, which is based on the Galerkin formulation and 1D-IRBFNs, for solving elliptic problems. From a Galerkin-approach point of view, it will be

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shown that the proposed technique has several advantages: (a) natural boundary conditions are forced to be satisfied exactly, (b) multiple boundary conditions are incorporated more efficiently. From a RBF-approach point of view, it will be shown that (a) the proposed method is capable of handling much larger data sets, (b) its accuracy is considerably better than that of the 1D-IRBFN collocation technique, (c) it is able to yield almost the same levels of accuracy for the solutions of Neumann- and Dirichlet-type problems. An additional attractiveness of the proposed technique is that it facilitates a higher-order continuity of the approximate solution across the subdomain interfaces.

The paper is organised as follows. Brief reviews of the Galerkin formulation and 1D-IRBFNs are given in Sections 2 and 3, respectively. The Galerkin 1D-IRBFN method is presented in Section 4, followed by several numerical examples in Section 5 to demonstrate the attractiveness of the proposed method. Section 6 concludes the paper.

2. Galerkin approach

The Galerkin-type approach is well documented in the literature. The reader is referred to, see, for example, [1,15,16], for a full comprehensive description. A brief review of this approach is given below.

Consider a boundary-value problem defined by a linear DE and its boundary conditions

$$L(\bar{u}) = 0, \quad \mathbf{x} \in \Omega, \tag{1}$$

$$B(\bar{u}) = 0, \quad \mathbf{x} \in \Gamma, \tag{2}$$

where $\bar{u}$ is the field/dependent variable (the overbar denotes the exact solution), L and B the prescribed known operators, Ω the domain of interest and Γ the boundaries of the domain Ω.

An approximate solution, denoted by u, to the set of (1) and (2) can be sought in the form

$$\bar{u}(\mathbf{x}) \approx u(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i \phi_i(\mathbf{x}), \tag{3}$$

where $\{\alpha_i\}_{i=1}^N$ is the set of unknown coefficients and $\{\phi_i(\mathbf{x})\}_{i=1}^N$ the set of linearly-independent functions. The terms ϕ_i are usually referred to as the trial/basis/approximating functions.

Assume that a function u is constructed to satisfy the DE (1) at every point on the domain Ω, it leads to

$$\int_\Omega wL(u)\,\mathrm{d}\Omega = 0, \tag{4}$$

for any function w that is bounded on Ω .

Similarly, assume that the function u also satisfies the boundary conditions (2), it follows that

$$\int_\Gamma \tilde{w}B(u)\,\mathrm{d}\Gamma = 0, \tag{5}$$

for any bounded function $\tilde{w}$. The functions w and $\tilde{w}$ are often referred to as the weighting/test functions.

Under assumptions (4) and (5), the approximate solution u is also the exact solution $\bar{u}$ itself, and the system defined by (1) and (2) is equivalent to the following integral statement:

$$\int_\Omega wL(u)\,\mathrm{d}\Omega + \int_\Gamma \tilde{w}B(u)\,\mathrm{d}\Gamma = 0, \tag{6}$$

which is satisfied for all bounded functions w and $\tilde{w}$.

However, in practice, one is able to employ finite sets of w and $\tilde{w}$ in the above integral statements, which result in an approximate solution.

If the weighting functions w and $\tilde{w}$ have sufficient degrees of continuity, integrations by parts can be applied to derivative terms in (6), leading to other integral statements, namely the weak and inverse forms, which can be expressed as

$$\int_\Omega C(w)D(u)\,\mathrm{d}\Omega + \int_\Gamma E(\tilde{w})F(u)\,\mathrm{d}\Gamma = 0, \tag{7}$$

where the order of continuity required for the u solution is reduced. One can thus use either (6) or (7) to determine the approximate solution u. These integral forms of weighted residuals will allow the approximation to be conducted subdomain by subdomain. Different types of w and $\tilde{w}$ will constitute different numerical approaches (e.g. point collocation, subdomain collocation and Galerkin-type ones). For the Galerkin-type approach, the weighting functions are chosen from the same set of functions as the trial functions. This approach usually leads to symmetric matrices.

3. One-dimensional integrated RBFNs

Consider a univariate function $f(x)$. The basic idea of the integral RBF scheme [17] is to decompose a pth-order derivative of the function f into RBFs

$$\frac{\mathrm{d}^p f(x)}{\mathrm{d}x^p} = \sum_{i=1}^{N} w_i g_i(x), \tag{8}$$

where $\{w_i\}_{i=1}^N$ is the set of network weights, and $\{g_i(x)\}_{i=1}^N$ the set of RBFs. For a convenient description of the integral scheme, we replace the notation $g_i(x)$ with the notation $I_i^{(p)}(x)$ that contains information about derivative order of f. By integrating (8), lower-order derivatives and the function itself are then obtained

$$\frac{\mathrm{d}^{p-1} f(x)}{\mathrm{d}x^{p-1}} = \sum_{i=1}^{N} w_i I_i^{(p-1)}(x) + c_1, \tag{9}$$

$$\frac{\mathrm{d}^{p-2} f(x)}{\mathrm{d}x^{p-2}} = \sum_{i=1}^{N} w_i I_i^{(p-2)}(x) + c_1 x + c_2, \tag{10}$$

$$\vdots$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \sum_{i=1}^{N} w_i I_i^{(1)}(x) + c_1 \frac{x^{p-2}}{(p-2)!} + c_2 \frac{x^{p-3}}{(p-3)!} + \cdots + c_{p-2}x + c_{p-1}, \tag{11}$$

$$f(x) = \sum_{i=1}^{N} w_i I_i^{(0)}(x) + c_1 \frac{x^{p-1}}{(p-1)!} + c_2 \frac{x^{p-2}}{(p-2)!} + \cdots + c_{p-1}x + c_p, \tag{12}$$

where $I_i^{(p-1)}(x) = \int I_i^{(p)}(x)\,\mathrm{d}x$, $I_i^{(p-2)}(x) = \int I_i^{(p-1)}(x)\,\mathrm{d}x, \ldots, I_i^{(0)}(x) = \int I_i^{(1)}(x)\,\mathrm{d}x$, and $\{c_1, c_2, \ldots, c_p\}$ are the constants of integration.

Unlike conventional differential schemes, the starting point of the integral scheme can vary in use, depending on the particular application under consideration. The scheme is said to be of order p, denoted by IRBFN-p, if the pth-order derivative is taken as the starting point.

Evaluation of (8)–(12) at a set of collocation points $\{x_j\}_{j=1}^N$ leads to

$$\widehat{\frac{\mathrm{d}^p f}{\mathrm{d}x^p}} = \widehat{\mathscr{I}}_{[p]}^{(p)} \widehat{\alpha}, \tag{13}$$

$$\widehat{\frac{\mathrm{d}^{p-1} f}{\mathrm{d}x^{p-1}}} = \widehat{\mathscr{I}}_{[p]}^{(p-1)} \widehat{\alpha}, \tag{14}$$

$$\vdots$$

$$\widehat{\frac{\mathrm{d}f}{\mathrm{d}x}} = \widehat{\mathscr{I}}_{[p]}^{(1)} \widehat{\alpha}, \tag{15}$$

$$\widehat{f} = \widehat{\mathscr{I}}_{[p]}^{(0)} \widehat{\alpha}, \tag{16}$$