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Numerical transfer-method with boundary conditions for arbitrary curved beam elements

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ABSTRACT

A novel numerical transfer-method is presented to solve a system of linear ordinary differential equations with boundary conditions. It is applied to determine the structural behaviour of the classical problem of an arbitrary curved beam element. The approach of this boundary value problem yields a unique system of differential equations. A Runge–Kutta scheme is chosen to obtain the incremental transfer expression. The use of a recurrence strategy in this equation permits to relate both ends in the domain where boundary conditions are defined. Semicircular arch, semicircular balcony and elliptic–helical beam examples are provided for validation.

1. Introduction

The problem to solve a system of linear ordinary differential equations with boundary conditions can be approach by using analytic or numerical strategies. Being not possible to always use an exact method, approximate procedures have been resorted to [1]. In last decades, several numerical methods have arisen to solve these boundary value problems.

It is the case of the shooting method [2], finite differences method [3], the spread use of finite element analysis [4] and the boundary element method in elasticity [5].

There exists much literature on arbitrary curved beam elements [6,7].

Traditionally, the laws governing the mechanical behavior of a curved warped beam (applying the Euler–Bernoulli and Timoshenko theories) are defined by static equilibrium and kinematics [8,9] or dynamic motion equations [10]. Some authors present this definition by means of compact energy equations [11–13]. These interpretations of structural behavior laws have permitted to reach accurate results, for only some types of beams: for example, a circular arch element loaded in plane [14–18] and loaded perpendicular to its plane [19], parabolic and elliptical beams loaded in plane [20–22] or a helix uniformly loaded [23].

In this article, the authors focus on the arbitrary curved beam model, by means of a unique system of twelve ordinary differential equations with boundary conditions [24].

The numerical method proposed is applied to the system of differential equations, obtaining an incremental equation based on the transfer matrix.

First-, second- and fourth-order Runge–Kutta approximations are adopted. Using the preceding incremental expression as a recurrence scheme, both extremes are related, reaching a system of algebraic equations with constant dimension regardless of the number of intervals. This boundary value problem is solved identifying the known support values in the above algebraic system. Hence, all the extremes values are determined and the recurrence scheme gives the solution at any point in the domain. Accurate results are provided in examples for validation.

2. The differential system

A curved beam is generated by a plane cross section which centroid sweeps through all the points of an axis line. The vector radius $\mathbf{r} = \mathbf{r}(s)$ expresses this curved line, where s (arc length of the centroid line) is the independent variable of the linear structural problem.

The reference system used to represent the intervening known and unknown functions is the Frenet frame. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are:

$$\mathbf{t} = D\mathbf{r}, \quad \mathbf{n} = \frac{D^2\mathbf{r}}{|D^2\mathbf{r}|}, \quad \mathbf{b} = \mathbf{t} \wedge \mathbf{n} \tag{1}$$

where $D = \mathrm{d}/\mathrm{d}s$ is the derivative with respect to the parameter s.

The natural equations of the centroid line are expressed by the flexion curvature $\chi = \sqrt{D^2\mathbf{r} \cdot D^2\mathbf{r}}$ and the torsion curvature $\tau = D\mathbf{r} \wedge \left(D^2\mathbf{r} \cdot D^3\mathbf{r}\right)/(D^2\mathbf{r} \cdot D^2\mathbf{r})$.

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The Frenet–Serret formulas are [25]:

$$\begin{aligned} D\mathbf{t} &= \chi\mathbf{n} \\ D\mathbf{n} &= -\chi\mathbf{t} + \tau\mathbf{b} \\ D\mathbf{b} &= -\tau\mathbf{n} \end{aligned} \tag{2}$$

Assuming the habitual principles and hypotheses (Euler–Bernuolli and Timoshenko classical beam theories) and considering the stresses associated with the normal cross-section (σ, τ_n, τ_b), the geometric characteristics of the section are: area $A(s)$, shearing coefficients $\alpha_n(s)$, $\alpha_{nb}(s)$, $\alpha_{bn}(s)$, $\alpha_b(s)$, and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$. Longitudinal $E(s)$ and transversal $G(s)$ elasticity moduli give the elastic condition of the material.

Applying equilibrium and kinematics laws in a differential element of the curve, the system of differential equations governing the structural behavior of the curved beam can be obtained [24]:

$$\begin{aligned}
& DN - \chi V_n + q_t = 0 \\
& \chi N + DV_n - \tau V_b + q_n = 0 \\
& \tau V_n + DV_b + q_b = 0 \\
& DT - \chi M_n + k_t = 0 \\
& -V_b + \chi T + DM_n - \tau M_n + k_n = 0 \\
& V_n + \tau M_n + DM_b + k_b = 0 \\
& -\frac{T}{GI_t} + D\phi - \chi\theta_n - \Theta_t = 0 \\
& -\frac{M_n I_b}{E\left[I_n I_b - I_{nb}^2\right]} - \frac{M_b I_{nb}}{E\left[I_n I_b - I_{nb}^2\right]} + \chi\phi + D\theta_n - \tau\theta_b - \Theta_n = 0 \\
& -\frac{M_n I_{nb}}{E\left[I_n I_b - I_{nb}^2\right]} - \frac{M_b I_n}{E\left[I_n I_b - I_{nb}^2\right]} + \tau\theta_n + D\theta_b - \Theta_b = 0 \\
& -\frac{N}{EA} + Du - \chi v - \Delta_t = 0 \\
& -\frac{\alpha_n V_n}{GA} - \frac{\alpha_{nb} V_b}{GA} - \theta_b + \chi u + Dv - \tau w - \Delta_n = 0 \\
& -\frac{\alpha_{bn} V_n}{GA} - \frac{\alpha_b V_b}{GA} + \theta_n + \tau v + Dw - \Delta_b = 0
\end{aligned} \tag{3}$$

The first six rows of the system Eq. (3) represent the equilibrium equations.

The functions involved in the equilibrium equation are:

- Internal forces:
  $N\mathbf{t} + V_n\mathbf{n} + V_b\mathbf{b} = \int_A \sigma\, dA\mathbf{t} + \int_A \tau_n\, dA\mathbf{n} + \int_A \tau_b\, dA\mathbf{b}$
- Internal moments:
  $T\mathbf{t} + M_n\mathbf{n} + M_b\mathbf{b} = \int_A (\tau_b n - \tau_n b)\, dA\mathbf{t} + \int_A \sigma b\, dA\mathbf{n} - \int_A \sigma n\, dA\mathbf{b}$
- Load force: $q_t\mathbf{t} + q_n\mathbf{n} + q_b\mathbf{b}$
- Load moment: $k_t\mathbf{t} + k_n\mathbf{n} + k_b\mathbf{b}$

The last six rows of the system Eq. (3) represent the kinematics equations.

- Rotations: $\phi\mathbf{t} + \theta_n\mathbf{n} + \theta_b\mathbf{b}$
- Displacements: $u\mathbf{t} + v\mathbf{n} + w\mathbf{b}$
- Rotation load: $\Theta_t\mathbf{t} + \Theta_n\mathbf{n} + \Theta_b\mathbf{b}$
- Displacement load: $\Delta_t\mathbf{t} + \Delta_n\mathbf{n} + \Delta_b\mathbf{b}$

The differential system (3) can be annotated in vector mode as follows:

$$D\mathbf{e}(s) = [\mathbf{T}_D(s)]\mathbf{e}(s) + \mathbf{q}(s) \tag{4}$$

where $\mathbf{e}(s) = \{N, V_n, V_b, T, M_n, M_b, \phi, \theta_n, \theta_b, u, v, w\}^{\mathrm{T}}$ is the state vector of internal forces and deflections at a point s of the beam element, named effect in the section,

$$[\mathbf{T}_D(s)] = \begin{bmatrix}
0 & \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\chi & 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\chi & 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{GI_t} & 0 & 0 & 0 & \chi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{I_b}{E[I_nI_b-I_{nb}^2]} & \frac{I_{nb}}{E[I_nI_b-I_{nb}^2]} & -\chi & 0 & \tau & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{I_{nb}}{E[I_nI_b-I_{nb}^2]} & \frac{I_n}{E[I_nI_b-I_{nb}^2]} & 0 & -\tau & 0 & 0 & 0 & 0 \\
\frac{1}{EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi & 0 \\
0 & \frac{\alpha_n}{GA} & \frac{\alpha_{nb}}{GA} & 0 & 0 & 0 & 0 & 0 & 1 & -\chi & 0 & \tau \\
0 & \frac{\alpha_{bn}}{GA} & \frac{\alpha_b}{GA} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\tau & 0
\end{bmatrix}$$

is the infinitesimal transfer matrix, and $\mathbf{q}(s) = \{-q_t, -q_n, -q_b, -k_t, -k_n, -k_b, \Theta_t, \Theta_n, \Theta_b, \Delta_t, \Delta_n, \Delta_b\}^{\mathrm{T}}$ is the applied load.

This general expression of the mechanical behavior of a curved element, which gives the relationship between the unknown effect $\mathbf{e}(s)$ (internal forces and displacements) produced by the load $\mathbf{q}(s)$ adopted, is a unique system of twelve linear ordinary differential equations.

3. Numerical transfer-method

3.1. Incremental transfer equation

Once the differential system Eq. (4) is introduced, the objective of this section is obtaining an incremental transfer equation that relates the state vector (effect in the section) at two consecutive points $\mathbf{e}(s_i)$ and $\mathbf{e}(s_{i+1})$ in the curve. This finite equation is determined using the first-, second- and fourth-order Runge–Kutta numerical approximation [26]. This relation constitutes the general term of the latter recurrence scheme.

3.1.1. First-order approximation

Applying the first-order approximation to Eq. (4), the effect vectors in two consecutive points of the beam element are related as follows:

$$\mathbf{e}(s_{i+1}) = \mathbf{e}(s_i) + \mathbf{k}_1 \Delta s \tag{5}$$

where

$$\mathbf{k}_1 = [\mathbf{T_D}(s_i)]\mathbf{e}(s_i) + \mathrm{q}(s_i)$$