Engineering Analysis with Boundary Elements 33 (2009) 1302–1315

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A practical examination of the errors arising in the direct collocation boundary element method for acoustic scattering

B.E. Treeby [a,*,1], J. Pan [b]

[a] Department of Medical Physics and Bioengineering, University College London, Gower Street, London WC1E 6BT, United Kingdom
[b] School of Mechanical Engineering, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia

ARTICLE INFO

Article history:
Received 2 December 2008
Accepted 25 June 2009
Available online 25 July 2009

Keywords:
Boundary element method
Non-linear coordinate transforms
Singular integrals
Integration error
Boundary singularities

ABSTRACT

For many engineers and acousticians, the boundary element method (BEM) provides an invaluable tool in the analysis of complex problems. It is particularly well suited for the examination of acoustical problems within large domains. Unsurprisingly, the widespread application of the BEM continues to produce an academic interest in the methodology. New algorithms and techniques are still being proposed, to extend the functionality of the BEM, and to compute the required numerical tasks with greater accuracy and efficiency. However, for a given global error constraint, the actual computational accuracy that is required from the various numerical procedures is not often discussed. Within this context, this paper presents an investigation into the discretisation and computational errors that arise in the BEM for acoustic scattering. First, accurate routines to compute regular, weakly singular, and nearly weakly singular integral kernels are examined. These are then used to illustrate the effect of the requisite boundary discretisation on the global error. The effects of geometric and impedance singularities are also considered. Subsequently, the actual integration accuracy required to maintain a given global error constraint is established. Several regular and irregular scattering examples are investigated, and empirical parameter guidelines are provided.

1. Introduction

The numerical solution of boundary integral equations via the boundary element method (BEM) has become a popular approach for solving acoustical problems. The now ubiquitous presence of high-speed computing has made it possible for engineers and acousticians to investigate any number of complex scenarios for which analytical solutions do not exist, or for which approximate solutions are not well suited. Unsurprisingly, the widespread application of the BEM continues to produce an academic interest in the methodology. New algorithms and techniques are still being proposed which extend the functionality of the BEM and allow the required numerical tasks to be computed with greater accuracy and efficiency. There are now a myriad of approaches for formulating the governing equations, discretising and computing the boundary quantities, computing singular integrals, and ensuring the uniqueness of the final solution. However, the overwhelming extent of the available literature paradoxically increases the difficulty of understanding the technique. Much of the advanced literature is focused on very specific aspects of the BEM and the consequence of utilising particular numerical routines or element types on the overall problem accuracy is not always immediately clear.

In an academic sense, formal mathematical investigations on the convergence of the various discretisation and numerical procedures lie within the (now) historical literature (e.g., [1,2]). In contrast, within engineering fields it is recurrently only the relative number of elements that is considered, with six elements per wavelength the most frequently prescribed guideline (see [3] and references therein). In fact, the errors in the BEM can arise from several distinct locations:

- the discrete approximation of the varying boundary quantities;
- the discrete approximation of the boundary geometry;
- geometric and impedance singularities;
- the numerical computation of regular integrals;
- the numerical computation of singular integrals;
- and, the solution of the system of equations.

The first three may be regarded as discretisation errors, the next two as quadrature errors, and the last incorporates the well-known non-uniqueness difficulty. Despite the continual advances in numerical and BEM literature, the actual accuracy that is

* Corresponding author. Tel.: +44 20 7679 7086; fax: +44 20 7679 0255.
E-mail address: btreeby@mpb.ucl.ac.uk (B.E. Treeby).
[1] This work was completed while at the School of Mechanical Engineering, The University of Western Australia.

0955-7997/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2009.06.005

required from the various computations to maintain a given level of global accuracy is rarely discussed. In this context, these errors are explored in detail here, primarily in relation to the direct collocation BEM for the two-dimensional (2D) Helmholtz equation. The investigation is practical in nature, rather than mathematically formal, and empirical guidelines are established for the various error constituents.

The basic formulation and use of the collocation BEM is well known (e.g., [4–7]). Arbitrary boundary surfaces are first discretised into a finite number of boundary elements. Across each element, approximations are made about the variation of the boundary quantities (the pressure and velocity in the direct BEM) and geometry. The boundary quantities are assigned at a discrete number of nodes per element (referred to as the element order), and shape functions are used to approximate the values at the other positions. The variation of the boundary geometry is modelled in the same way. The overall computational effort is directly related to the total number of boundary nodes (often referred to as the number of system degrees of freedom).

Considering first only the error associated with the discretised approximation of the varying boundary quantities, the solution convergence with various mesh modifications is well documented (e.g., [8]). If the number of elements is increased without modifying the number of nodes per element (the so-called refinement or h-method), an algebraic convergence related to the element order is obtained as the total number of degrees of freedom is increased. If the number of elements is kept constant and instead the number of nodes per element is increased (the so-called enrichment or p-method), an exponential convergence is obtained. However, if the boundary is not smooth, i.e., it contains geometric or impedance singularities, the convergence of the p-method degenerates. Boundary singularities occur because of abrupt changes or discontinuities in the surface profile or its properties. These cause the solution and its derivatives to vary rapidly or become unbounded [9]. A combination of the refinement method near boundary singularities and the enrichment method elsewhere (the so-called hp-method) can restore the exponential convergence of the p-method [10,11].

In addition to the number of elements (and nodes per element), the continuity of the quantities around the boundary, enforced by the element type, is also of importance. If the outer nodes on each element are placed at the endpoints and shared between adjoining elements, the surface quantities remain continuous around the boundary. Continuous isoparametric linear and parabolic elements are the most widely used in engineering applications [3]. Alternatively, if the outer nodes are located within the element, the boundary quantities become discontinuous. Compared to continuous elements, this requires an increase in the overall number of nodes to describe the same order of boundary variation (as nodes are no longer shared between elements). Despite this, if the nodes are positioned correctly, discontinuous elements can provide an increase in accuracy for the same number of system degrees of freedom [12]. For three-dimensional quadrilateral or triangular elements, the zeros of the Legendre polynomials provide this optimum position. Discontinuous elements also have the distinct advantage of the normal direction always being well defined. Similarly, they naturally satisfy the C^1 continuity condition.

In contrast to the approximation of the boundary quantities, the variation of the boundary geometry should remain continuous from one element to the next. For discontinuous boundary quantities, this requires the use of continuous shape functions based on a different set of nodal values. However, this does not add any particular computational complexity, as the geometric coordinates of the utilised nodes are required in any case. Rather, the inherent separation of the boundary geometry and quantities makes it straightforward to use non-isoparametric elements. This is computationally efficient when using higher-order shape functions to describe the variation of the boundary quantities for simple boundary geometries. The requirement for the geometric discretisation is that the variation of the boundary elements matches that of the underlying problem. For isoparametric linear elements, the error from the geometric discretisation of curved boundaries is of the same order as that from the discretisation of the boundary quantities [8].

Returning to the convergence of the BEM, rather than simply uniformly increasing the total number of elements or nodes to improve accuracy, local error indicators can be used to specify regions where the boundary discretisation must be refined [13]. The use of suitable adaptive methods can significantly improve the rate of convergence, as degrees of freedom automatically become concentrated near boundary singularities and other areas where the surface quantities vary rapidly [13–16]. However, the computation of many error indicators (upon which the adaptive procedures rely) is very expensive; on the order of the BEM analysis itself [17]. Moreover, at each step of the mesh refinement, the problem must be recomputed. As an alternative, boundary meshes may be graded *a priori* so that the optimal values of convergence are maintained [11,18]. Similarly, boundary singularities may also be counteracted by using specially designed interpolation functions [8].

Apart from discretisation errors, the computation of the element-wise boundary integration (typically via numerical quadrature) may also introduce errors into the BEM. In a commercial BEM package, the modification of the quadrature routines is unlikely to be facilitated. Moreover, in classical studies, the quadrature error is rarely considered important in comparison with that from the boundary discretisation (e.g., [19]). However, care must be taken that it is sufficiently small [20]. For regular integrals, this constraint is of no particular consequence as standard quadrature routines are able to compute the required integration tasks very accurately. Nonetheless, it is important to ensure that singular and nearly singular integrals can be computed with the same accuracy. This is particularly important for boundaries that yield coefficient matrices where the diagonal (singular) terms are dominant. Although integration errors can generally be reduced by increasing the order of the quadrature routines, doing so arbitrarily without consideration of the actual accuracy requirements can result in a significant computational penalty.

The final source of error relates to the solution of the system of equations for the unknown boundary quantities. For a well-conditioned coefficient matrix, this error is negligible. However, at certain characteristic frequencies that depend on the problem geometry, the solution may no longer be unique. This non-uniqueness has no physical analogy and is simply a mathematical artefact of the integral formation; the objective problem itself has a unique solution. These (so-called) characteristic frequencies occur at the resonant frequencies of the corresponding interior Dirichlet problem (the non-uniqueness problem occurs because the interior and exterior operators are adjoint [21]). The most mathematically robust way to overcome this difficulty is to use the Helmholtz integral equation in combination with its normal derivative. The latter conversely suffers from non-uniqueness at the resonant frequencies of the interior Neumann problem. Burton and Miller showed that the two integral equations share only one common solution and thus their combination will overcome the mathematical uniqueness difficulty [22]. However, the normal derivative integral equation contains a hyper-singular integral kernel which is numerically formidable (the integral does not exist because of divergent terms in the primitive function of the