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Numerical solution for degenerate scale problem for exterior multiply connected region

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ABSTRACT

Based on some previous publications, this paper investigates the numerical solution for degenerate scale problem for exterior multiply connected region. In the present study, the first step is to formulate a homogenous boundary integral equation (BIE) in the degenerate scale. The coordinate transform with a magnified factor, or a reduced factor h is performed in the next step. Using the property $\ln(hx)=\ln(x)+\lg(h)$, the new obtained BIE equation can be considered as a non-homogenous one defined in the transformed coordinates. The relevant scale in the transformed coordinates is a normal scale. Therefore, the new obtained BIE equation is solvable. Fundamental solutions are introduced. For evaluating the fundamental solutions, the right-hand terms in the non-homogenous equation, or a BIE, generally take the value of unit or zero. By using the obtained fundamental solutions, an equation for evaluating the magnified factor "h" is obtained. Finally, the degenerate scale is obtainable. Several numerical examples with two ellipses in an infinite plate are presented. Numerical solutions prove that the degenerate scale does not depend on the normal scale used in the process for evaluating the fundamental solutions.

1. Introduction

The boundary integral equation (BIE) was widely used in elasticity, and the fundamental for BIE could be found from [1–3]. Heritage and early history of the boundary element method was summarized more recently [4].

The degenerate scale problem in BIE is a particular boundary value problem in plane elasticity as well as in Laplace equation. The problem typically arises in the exterior Dirichlet problem of plane elasticity. For example, an infinite plate contains an elliptic notch with assumption of vanishing displacements on the elliptic contour. In this case, if the size reaches a critical size, there is a stress field existing in the notched infinite region. The critical size is called the degenerate scale. This situation means that the homogenous BIE has been shown to yield a non-trivial solution for boundary tractions, (or $\sigma_{ij}\neq 0$ at the boundary points) when the adopted configuration is equal to a degenerate scale. From the viewpoint of engineering, the non-trivial solution of the homogenous BIE is an illogical one. Clearly, the relevant non-homogenous integral equation must have non-unique, or many solutions. Therefore, one must avoid meeting illogical solution caused by occurrence of the degenerate scale.

For the Laplace equation, the non-unique solution of a circle with a unit radius has been noted [5]. In fact, this is the degenerate scale problem. Some researchers define the degenerate scale problem such that the degenerate scale results in a zero eigenvalue for the integral operator.

Since the displacement and stress condition at infinity may have the following properties, $u_i=O(\ln R)$, $\sigma_{ij}=O(1/R)$, where R is the radius of a sufficient large circle, it is expected to have a non-trivial solution from the viewpoint of mathematics and mechanics [6].

Many researchers using a variety of methods studied the degenerate scale problems. Some researchers investigated the problem from mathematical theory of BIE. For example, it was pointed out that an explicit equation for evaluation of critical scales for a given boundary, when the single-layer operator fails to be invertible, is deduced. It is proved that there are either two simple critical scales or one double critical scale for any domain boundary [7]. The logarithmic function appearing in the integral kernel leads to the possibility of this operator being non-invertible, the solution of the BIE either being non-unique or not existing [8].

The boundary integral equation is used for a ring region with the vanishing displacements along the boundary. In some particular scale of the configuration, the corresponding homogenous equation has non-trivial solution for the boundary tractions [9,10]. Numerical technique for evaluating the degenerate scale was also suggested by using two sets of some particular solution based on a modified BIE [8]. Mathematical

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analysis of the degenerate scale problems for an elliptic-domain problem in elasticity was presented [11].

Numerical procedure was developed to evaluate the degenerate scale directly from the zero value of a determinant. In the procedure, the influence matrix is denoted by $U(a)$, where "a" is a size. In the computation, the size "a" is changed gradually. If at some particular value $a=\lambda$, the following conditions, det $U(a)|_{a=\lambda-\varepsilon}>0$ and det $U(a)|_{a=\lambda+\varepsilon}>0$ ($\varepsilon=0.00002$ a small value) are satisfied, the obtained $a=\lambda$ is the degenerate scale. In the time of evaluating the degenerate scale, the relevant non-trivial solution was also obtained [12,13].

It was pointed out that the influence matrix of the weakly singular kernel (logarithmic kernel) might be singular for the Dirichlet problem of Laplace equation when the geometry is reaching some special scale [14]. In this case, one may obtain a non-unique solution. The non-unique solution is not physically realizable. From the viewpoint of linear algebra, the problem also originates from rank deficiency in the influence matrices [14,15]. By using a particular solution in the normal scale, the degenerate scale can be evaluated numerically [14]. The degenerate scale problem in BIE for the two-dimensional Laplace equation was studied by using degenerate kernels and Fourier Series [16]. In the circular domain case, the degenerate scale problem in BIE for the two-dimensional Laplace equation was studied by using degenerate kernels and circulants, where the circulants mean that the influence matrices for the discrete system has a particular character [17].

Degenerate scale for multiply connected Laplace problems was studied [18]. It is concluded from the paper that: "no matter how many inner holes are randomly distributed inside the outer boundary, the Dirichlet problem with radius ($a_1=1.0$) of the outer boundary is not solvable due to the rank-deficiency matrix of $[U]$ in Eq. (13)".

The coordinate transform technique for evaluating the degenerate scale in for Laplace equation was proposed [5,14]. Similar derivation for the case of plane elasticity was studied [8]. Based on the mentioned references, this paper investigates the numerical solution for degenerate scale problem for exterior multiply connected region.

In the present study, the first step is to formulate a homogenous equation in the degenerate scale. The coordinate transform with a magnified factor, or a reduced factor "h" is performed in the next step. Using the property $\ln(hx)=\ln(x)+\lg(h)$, the new obtained BIE equation is non-homogenous defined in the transformed coordinates. The relevant scale in the transformed coordinates is a normal scale. Therefore, the new obtained BIE equation is solvable. Fundamental solutions are introduced. For evaluating the fundamental solutions, the right-hand terms in the non-homogenous equation, or a BIE, generally take the value of unit or zero. By using the obtained fundamental solutions, an equation for evaluating the magnified factor "h" is obtained. Finally, the degenerate scale is obtainable. Several numerical examples with two ellipses in an infinite plate are presented.

2. Formulation of the degenerate scale problem for exterior multiply connected region

The concept of using the coordinate transform technique to evaluate the degenerate scale was introduced [5]. This technique was used in the case of the Laplace equation [14]. Similar derivation for the case of plane elasticity was studied [8]. Based on the mentioned references, this paper investigates the numerical solution for degenerate scale problem for exterior multiply connected region.

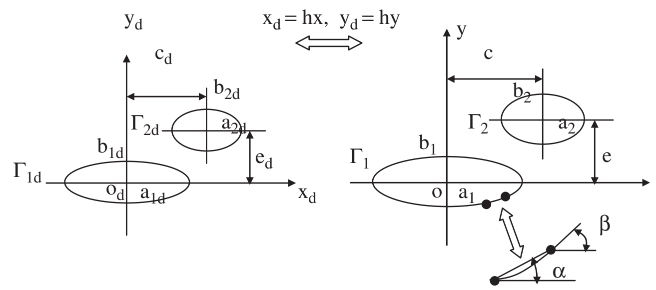

Fig. 1. Formulation of the degenerate scale problem, (a) the degenerate scale and (b) the normal scale with the relation to the degenerate scale by $x_d=hx$, $y_d=hy$, $a_{1d}=ha_1$, $b_{1d}=hb_1$, $a_{2d}=\delta a_2$, $b_{2d}=hb_2$, $c_d=hc$ and $e_d=he$.

In the first step, the degenerate scale problem for two holes is reduced to find a non-trivial solution for the following homogenous equation in the degenerate scale or in the coordinates $o_dx_dy_d$ (Fig. 1)

$$\int_{\Gamma_{1d}} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_{2d}} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=0,\ (\xi\in\Gamma_{1d},\ i=1,2) \tag{1}$$

$$\int_{\Gamma_{1d}} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_{2d}} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=0,\ (\xi\in\Gamma_{2d},\ i=1,2) \tag{2}$$

where Γ_{1d} and Γ_{2d} denotes the boundary of two ellipses, and the kernel $U_{ij}{}^*(\xi,x)$ is defined by

$$U^*_{ij}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}-0.5\delta_{ij}\} \tag{3}$$

where Kronecker deltas δ_{ij} is defined as, $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$, and

$$r_{,1}=\frac{x_1-\xi_1}{r}=\cos\alpha,\ r_{,2}=\frac{x_2-\xi_2}{r}=\sin\alpha$$
$$n_1=-\sin\beta,\ n_2=\cos\beta \tag{4}$$

where the angles α and β are indicated in Fig. 1. In Eq. (3), G denotes the shear modulus of elasticity and ν the Poisson's ratio. It was proved previously that the following kernel $U_{ij}{}^*(\xi,x)$ satisfies the regularity condition at infinity in BIE [12]. Therefore, this kernel is used in the formulation of the degenerate scale problem.

The equation may be solved in the normal scale with coordinates oxy. The two coordinates have the following relations (Fig. 1)

$$x_d=hx,\ y_d=hy,\ a_{1d}=ha_1,\ b_{1d}=hb_1,\ a_{2d}=\delta a_2,\ b_{2d}=hb_2,$$
$$c_d=hc,\ e_d=he \tag{5}$$

where "h" denotes a magnified factor or a reduced factor.

Therefore, Eqs. (1) and (2) can be converted to the following non-homogenous equations in the normal oxy coordinates

$$\int_{\Gamma_1} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_2} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=\frac{1}{\gamma}q_i,\ (\xi\in\Gamma_1,\ i=1,2) \tag{6}$$

$$\int_{\Gamma_1} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_2} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=\frac{1}{\gamma}q_i,\ (\xi\in\Gamma_2,\ i=1,2) \tag{7}$$

where Γ_1, Γ_2 denote relevant configuration in the normal scale and

$$\frac{1}{\gamma}=\frac{3-4\nu}{8\pi(1-\nu)G}\ln h \tag{8}$$