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Meshless analysis of two-dimensional Stokes flows with the Galerkin boundary node method

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ABSTRACT

In this paper, the Galerkin boundary node method (GBNM) is developed for the solution of stationary Stokes problems in two dimensions. The GBNM is a boundary only meshless method that combines a variational form of boundary integral formulations for governing equations with the moving least-squares (MLS) approximations for construction of the trial and test functions. Boundary conditions in this approach are included into the variational form, thus they can be applied directly and easily despite the MLS shape functions lack the property of a delta function. Besides, the GBNM keeps the symmetry and positive definiteness of the variational problems. Convergence analysis results of both the velocity and the pressure are given. Some selected numerical tests are also presented to demonstrate the efficiency of the method.

1. Introduction

Meshless (or meshfree) methods for numerical solutions of boundary value problems have generated much attention in recent years [1,2]. As opposed to the finite element method (FEM) and the boundary element method (BEM), the main feature of this type of method is the absence of an explicit mesh, and the approximate solutions are constructed entirely based on a cluster of scattered nodes. Although many kinds of meshless methods have been proposed, these methods can be simply divided into domain type and boundary type. Several domain type meshless methods, such as the element free Galerkin method (EFGM) [3], the generalized FEM [4], the $h-p$ meshless method [5,6], the reproducing kernel particle method [1], the moving least-square (MLS) reproducing kernel method [1], the reproducing kernel element method [1] and the finite point method [7] are very promising methods, and their mathematical foundations were well investigated.

Boundary integral equations (BIEs) are attractive computational techniques for linear and exterior problems as they can reduce the dimensionality of the original problem by one. Especially for exterior problems, the use of domain type methods requires discretization of the entire exterior, whereas with BIEs only the surface needs to be discretized. The boundary type meshless methods are developed by the combination of the meshless idea with BIEs, such as the boundary node method (BNM) [8], the boundary cloud method [9], the hybrid boundary node method [10], the boundary point interpolation method [11], the boundary element-free method [12] and the Galerkin boundary node method (GBNM) [13]. Compared with the domain type meshless methods, they require only a nodal data structure on the bounding surface of a body whose dimension is less than that of the domain itself. So like the BEM, they are superior in treating problems dealing with infinite or semi-infinite domains. Nevertheless, most boundary type meshless methods found in the literature lack a rich mathematical background to justify their use.

The BNM is formulated using the MLS approximations and the technique of BIEs. This method exploits the dimensionality of BIEs and the meshless attribute of the MLS. However, since the MLS approximations lack the delta function property, the BNM cannot exactly satisfy boundary conditions. This issue becomes even more severe in the BNM because a large number of boundary conditions need to be satisfied. The strategy employed in the BNM [8] involves a new definition of the discrete norm used for the generation of the MLS approximations, which doubles the number of system equations.

Based on the BNM, the GBNM is proposed by Li et al. [13], which has combined the MLS technique and an equivalent variational form of BIEs. The MLS approximations are used to generate the trial and test functions. In contrast with the BNM, boundary conditions in the GBNM can be implemented directly and easily via multiplying the MLS shape function and integrating over the boundary. Besides, the GBNM keeps the symmetry and positive definiteness of the variational problems, a property that makes the method an ideal choice for coupling the FEM or other established meshless methods such as the EFGM. The GBNM has

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been applied to problems of Laplace equation [13] and biharmonic equation [14], and for Stokes equation with slip boundary conditions [15].

In this paper, the GBNM is further developed for solving the 2D Stokes equation with no-slip boundary conditions. The Stokes problem has been usually applied to model incompressible creeping flows where the fluid Reynolds number is very low. This problem is the first step in order to consider the nonlinear Navier–Stokes equations of incompressible fluids. BIEs have been successfully implemented for the numerical computation of Stokes problems [16–22]. Unlike domain type methods, the incompressibility condition is automatically satisfied when BIEs are used, so BIEs-based methods are well suited in solving the Stokes problem especially for the exterior problem.

As in many other meshless methods such as the EFGM and the BNM, background cells are used in the GBNM for numerical integration over the boundary. Cells are used for integration only, and have no restriction on shape or compatibility. The topology of cells can be much simpler than that of elements in the BEM or the FEM, since cells can be divided into smaller ones without affecting their neighbors in any way—such is not the case with boundary or finite elements. This feature makes meshless methods especially suited for adaptive techniques [6,23]. In the case of the cell structure is coincided with the boundary, error estimates of the GBNM have been established in Sobolev spaces for problems in potential theory [13,14] and fluid [15,24]. In general, as the element in the BEM, the cell structure is an approximation of the boundary. When there exists difference between the cell structure and the boundary, the error results from the approximation of the boundary by cells needs to be considered. In this paper, we give the optimal asymptotic error estimates of the GBNM for solving Stokes problems in Sobolev spaces.

The following discussions begin with the brief description of the MLS approximation in Section 2. The formulations of the GBNM for Stokes flows are developed in Section 3. Error estimates are established in Section 4. Numerical examples are presented in Section 5. Section 6 contains some conclusions.

2. The MLS approximation scheme

2.1. Notations

Let Ω be a bounded domain in $\mathbb{R}^2$ of points $\mathbf{x}=(x_1,x_2)$, its boundary Γ assumed to be sufficiently smooth, and let Ω' be the complementary of $\overline{\Omega}=\Omega+\Gamma$. For any point $\mathbf{x}\in\Gamma$, we use $\Re(\mathbf{x})$ to denote the domain of influence of $\mathbf{x}$. Let $Q_N=\{\mathbf{x}_i\}_{i=1}^N$ be an arbitrarily chosen set of N boundary nodes $\mathbf{x}_i\in\Gamma$. The set Q_N is used for defining a finite open covering $\{\Re_i\}_{i=1}^N$ of Γ composed of N balls $\Re_i$ centered at the points $\mathbf{x}_i$, $i=1,2,\ldots,N$, where $\Re_i=\Re(\mathbf{x}_i)$ is the influence domain of $\mathbf{x}_i$. Assume that $\kappa(\mathbf{x})$ boundary nodes lie on $\Re(\mathbf{x})$. Then, we use the notation $I_1,I_2,\ldots,I_\kappa$ to express the global sequence number of these nodes, and define $\wedge(\mathbf{x}):=\{I_1,I_2,\ldots,I_\kappa\}$. Besides, we use

$$\Re^i:=\{\mathbf{x}\in\Gamma:\mathbf{x}_i\in\Re(\mathbf{x})\},\quad 1\le i\le N \tag{1}$$

to denote the set of boundary points whose influence domain includes the boundary node $\mathbf{x}_i$. For different boundary point $\mathbf{x}$, because $\Re(\mathbf{x})$ varies from point to point, $\Re^i\equiv\Re_i$ if and only if the radii of $\Re(\mathbf{x})$ is a constant for any $\mathbf{x}\in\Gamma$.

Let L be the length of Γ, then Γ is a curve having L-periodic parametric representation with respect to the arc-length s,

$$\mathbf{x}=X(s),\quad s\in[0,L] \tag{2}$$

Obviously, $X(s)$ is a mapping from $\mathbb{R}$ onto $\mathbb{R}^2$. Denoted by ℓ the continuous order of Γ, then $X(s)\in(C^\ell)^2$ and $\partial^m X(s)/\partial s^m$ is bounded provided that $m\le\ell$. From Eq. (2), boundary nodes $\mathbf{x}_i$ can be represented as $\mathbf{x}_i=X(s_i)$, $1\le i\le N$. Since Γ is closed, we set $s_0=s_N-L$, then $\mathbf{x}_0=\mathbf{x}_N$. Let

$$h:=\max_{1\le i\le N}(s_i-s_{i-1}),\quad h_i:=|\overrightarrow{\mathbf{x}_{i-1}\mathbf{x}_i}|=|X(s_i)-X(s_{i-1})| \tag{3}$$

then from the fact that $\partial X(s)/\partial s$ is bounded we have

$$h_i=O(h) \tag{4}$$

Consequently, the parameter h can be used to measure the nodal spacing.

2.2. The MLS technique

Assume that $\mathbf{x}\in\Gamma$, the MLS approximation for a given function v is defined as [13]

$$v(\mathbf{x})\approx\mathscr{M}v(\mathbf{x})=\sum_{i=1}^N\Phi_i(\mathbf{x})v_i \tag{5}$$

where $\mathscr{M}$ is an approximation operator, and

$$\Phi_i(\mathbf{x})=\Phi_i(X(s))=\begin{cases}\sum_{j=0}^{\overline{m}}P_j(s)[\mathbf{A}^{-1}(s)\mathbf{B}(s)]_{jk}, & i=I_k\in\wedge(\mathbf{x}),\\ 0, & i\notin\wedge(\mathbf{x}),\end{cases}\quad 1\le i\le N \tag{6}$$

and the matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ being defined by

$$\mathbf{A}(s)=\sum_{k\in\wedge(X(s))}w_k(s)\mathbf{P}(s_k)\mathbf{P}^{\mathrm{T}}(s_k) \tag{7}$$

$$\mathbf{B}(s)=[w_{I_1}(s)\mathbf{P}(s_{I_1}),w_{I_2}(s)\mathbf{P}(s_{I_2}),\ldots,w_{I_\kappa}(s)\mathbf{P}(s_{I_\kappa})] \tag{8}$$

in which $\mathbf{P}(s)$ is a vector of the polynomial basis, $\overline{m}+1$ is the number of monomials in the polynomial basis, w_k denote nonnegative weight functions which belong to $C_0^\alpha(\Re_i)$, $\alpha\ge 0$, and satisfy $\sum_{k\in\wedge(s)}w_k(s)=1$.

For error analysis, we impose the following conditions:

Assumption 1. There is a nonnegative integer $\gamma\le\ell$ such that $\Phi_i(\mathbf{x})\in C^\gamma(\Gamma)$.

Assumption 2. There are positive integers $K_1(\mathbf{x})\ge\overline{m}$ and $K_2(\mathbf{x})$ such that for any $\mathbf{x}\in\Gamma$, there are at least $K_1(\mathbf{x})$ boundary nodes, and at most $K_2(\mathbf{x})$ boundary nodes lie on $\Re(\mathbf{x})$.

Assumption 3. There exist constants C_{w1} and C_{w2} independent of h such that $C_{w1}h^{-j}\le\|\partial^j w_i(s)\|_{L^\infty([0,L])}\le C_{w2}h^{-j}$, $0\le j\le\gamma$, $1\le i\le N$.

From Eq. (4) and Assumption 2, it can be verified that the radii of any boundary point's influence domain can be measured by the parameter h.

Proposition 2.1 (*Li and Liu [1]*). $\sum_{i\in\wedge(\mathbf{x})}\Phi_i(\mathbf{x})=1$.

Proposition 2.2 (*Li and Zhu [13]*). $\Phi_i(\mathbf{x})\in C_0^\gamma(\Re^i)$, $1\le i\le N$.

Theorem 2.1 (*Li and Zhu [13]*). *For any $v(\mathbf{x})\in H^{m+1}(\Gamma)$, there exists a constant C independent of h such that*

$$\|v(\mathbf{x})-\mathscr{M}v(\mathbf{x})\|_{H^k(\Gamma)}\le Ch^{m+1-k}\|v(\mathbf{x})\|_{H^{m+1}(\Gamma)},\quad 0\le k\le m\le\gamma \tag{9}$$

where $H^\tau(\Gamma),\tau\in\mathbb{R}$, denotes the Sobolev spaces of functions defined on Γ [20,25].