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# Solution method of interface cracks in three-dimensional transversely isotropic piezoelectric bimaterials

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**Abstract**

An analysis method is proposed for planar interface cracks of arbitrary shape in three-dimensional transversely isotropic piezoelectric bimaterials based on the analogy between the hyper-singular boundary integral–differential equations for interface cracks in purely elastic media and those in piezoelectric media with the electrically impermeable crack condition. The poling direction is along the $z$-axis of the Cartesian coordinate system and perpendicular to the interface. The singular indexes and the singular behaviors of the near crack-tip fields are studied. The results show that the extended stress $\sigma_{zz}-c_2D_z$ has the classical singularity $r^{-1/2}$, while the extended stress $\sigma_{zz}+c_4D_z$ possesses the well-known oscillating singularity $r^{-1/2\pm i\varepsilon}$ or the non-oscillating singularity $r^{-1/2\pm\kappa}$, where $\sigma_{zz}$ and $D_z$ are, respectively, the stress and electric displacement components, and $c_2$ and $c_4$ are two material constants. The three-dimensional transversely isotropic piezoelectric bimaterials are categorized into two groups, i.e., $\varepsilon$-group with non-zero value of $\varepsilon$ and $\kappa$-group with non-zero value of $\kappa$. Two new extended stress intensity factors $K_{I1}$ and $K_{I2}$ corresponding, respectively, to the extended stresses $\sigma_{zz}-c_2D_z$ and $\sigma_{zz}+c_4D_z$ are defined for interface cracks in three-dimensional transversely isotropic piezoelectric bimaterials. The material related constants including $\varepsilon$ or $\kappa$ for 15 bimaterials are calculated. The extended intensity factor of a penny-shaped interface crack is presented as an application of the proposed method.


*Keywords:* Piezoelectric bimaterial; Interface crack; Displacement discontinuity; Boundary integral equation; Intensity factor; Penny-shaped crack

## 1. Introduction

Due to the coupling effect between the mechanical and electric properties, piezoelectric material has been widely used in intelligent structures and systems. Multi-layered structures are often used to utilize the accumulative results of stacks to enhance the efficiency and sensitivity. Interfacial fracture is one of the major failure modes in these structures. The analysis of interface cracks in piezoelectric media has been drawing much attention [1–8]. The state of the art till 2001 can be found in the review paper by Zhang et al. [9]. In this field, one of the most interesting findings is that beside the classical singularity $r^{-1/2}$ and the well-known oscillatory singularity $r^{-1/2\pm i\varepsilon}$, the extended stresses have a new type of singularity $r^{-1/2\pm\kappa}$ near the crack tip in piezoelectric bimaterials. In recent years, new developments have been made, e.g., see [10–13]. It is found that transversely isotropic bimaterials with an impermeable interface crack are divided into two classes corresponding to the vanishing of the two singularity parameters $\varepsilon$ or $\kappa$. For three-dimensional interfacial cracks in piezoelectric media, however, only a few results are available due to the complexity of the problem. Tian and Rajapakse [14] discussed the axi-symmetric problem of an interfacial penny-shaped crack in a piezoelectric bimaterial system. Zhao et al. [15] derived the hyper-singular boundary integral–differential equations for planar interface cracks of arbitrary shape in three-dimensional two-phase transversely isotropic piezoelectric media and the boundary element method was used to solve these equations numerically.

The objective of this paper is to present a method to analyze planar interfacial cracks of arbitrary shape in

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three-dimensional piezoelectric bimaterials under electrically impermeable boundary condition and to find and characterize the singular behaviors of near crack-tip fields.

## 2. Boundary integral–differential equations for interfacial cracks

Consider a three-dimensional two-phase transversely isotropic piezoelectric medium with the interface being parallel to the plane of isotropy. A Cartesian coordinate system is set up such that the $xoy$-plane lies in the interface. There is a planar interface crack $S$ of arbitrary shape in the interface of the bimaterial as shown in Fig. 1. The upper and lower surfaces of $S$ are denoted by $S^+$ and $S^-$, respectively. The outer normal vectors of $S^+$ and $S^-$ have the relation

$$\{n_i\}|_{S^+} = -\{n_i\}|_{S^-} = \{0, 0, -1\}. \tag{1}$$

On the upper and lower crack faces the applied extended tractions satisfy the conditions

$$\begin{aligned} p_i(x, y, 0^+) &= -p_i(x, y, 0^-) = p_i(x, y), \\ &\quad i = (x, y, z \quad \text{or} \quad 1, 2, 3), \\ \omega(x, y, 0^+) &= -\omega(x, y, 0^-) = \omega(x, y). \end{aligned} \tag{2}$$

The boundary integral–differential equations for the interface crack are given [15]

$$\int_{S^+} [K_{zz1}||w|| + K_{z1}||\varphi||]\frac{1}{r^3}\,\mathrm{d}S(\xi,\eta) + 2\pi K_1\left[\frac{\partial||u||}{\partial x} + \frac{\partial||v||}{\partial y}\right] = -p_z(x, y), \tag{3}$$

$$\int_{S^+} [K_{zz2}||w|| + K_{z2}||\varphi||]\frac{1}{r^3}\,\mathrm{d}S(\xi,\eta) + 2\pi K_2\left[\frac{\partial||u||}{\partial x} + \frac{\partial||v||}{\partial y}\right] = -\omega(x, y), \tag{4}$$

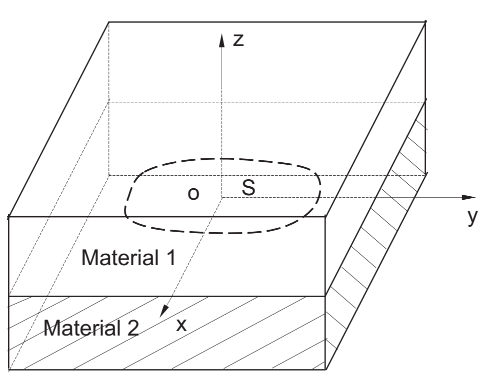


Fig. 1. An arbitrarily shaped planar interface crack $S$ in the $oxy$ plane in a piezoelectric bimaterial.

$$\begin{aligned} &\int_{S^+} \{[K_{zr3}\cos^2\theta + K_{z\theta3}\sin^2\theta||u|| + [(K_{zr3} - K_{z\theta3}) \\ &\quad \times \sin\theta \times \cos\theta]||v||\}\frac{1}{r^3}\,\mathrm{d}S(\xi,\eta) + 2\pi K_r\frac{\partial||w||}{\partial x} \\ &\quad + 2\pi K_\theta\frac{\partial||\varphi||}{\partial x} = -p_x(x, y), \end{aligned} \tag{5}$$

$$\begin{aligned} &\int_{S^+} \{[K_{zr3}\sin^2\theta + K_{z\theta3}\cos^2\theta||v|| + [(K_{zr3} - K_{z\theta3}) \\ &\quad \times \sin\theta \times \cos\theta]||u||\}\frac{1}{r^3}\,\mathrm{d}S(\xi,\eta) + 2\pi K_r\frac{\partial||w||}{\partial y} \\ &\quad + 2\pi K_\theta\frac{\partial||\varphi||}{\partial y} = -p_y(x, y), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &r^2 = (x - \xi)^2 + (y - \eta)^2, \quad \cos\theta = (\xi - x)/r, \\ &\quad \sin\theta = (\eta - y)/r, \end{aligned} \tag{7}$$

$\|u\|$, $\|v\|$, and $\|w\|$ are the displacement discontinuities, respectively, along the $x$-, $y$- and $z$-axes, and $\|\varphi\|$ the electric potential discontinuity across the crack faces, which are called the extended displacement discontinuities and are given by

$$\begin{aligned} &||u(x, y)|| = u(x, y, 0^+) - u(x, y, 0^-), \\ &||v(x, y)|| = v(x, y, 0^+) - v(x, y, 0^-), \\ &||w(x, y)|| = w(x, y, 0^+) - w(x, y, 0^-), \\ &||\varphi(x, y)|| = \varphi(x, y, 0^+) - \varphi(x, y, 0^-). \end{aligned} \tag{8}$$

In Eqs. (3)–(6), $K_{zz1}$, $K_{z1}$, $K_1$, $K_{zz2}$, $K_{z2}$, $K_2$, $K_{zr3}$, $K_{z\theta3}$, $K_r$ and $K_\theta$ are material related constants given in Appendix A. It can easily be seen that the kernels in Eqs. (3)–(6) have the singularity $r^{-3}$, and hence the integral–differential equations are hyper-singular.

## 3. Solution method of the integral equation

### 3.1. Solution method for the extended displacement discontinuity $||w|| + c_1||\varphi||$

Combining Eqs. (3) and (4), one obtains

$$\left(K_{zz1} - \frac{K_1 K_{zz2}}{K_2}\right)\int_{S^+} [||w|| + c_1||\varphi||]\frac{1}{r^3}\,\mathrm{d}S = -p_z + (K_1/K_2)\omega, \tag{9}$$

where

$$c_1 = \frac{K_{z1} - (K_1/K_2)K_{z2}}{K_{zz1} - (K_1/K_2)K_{zz2}}. \tag{10}$$

Eq. (9) is the hyper-singular boundary integral equation for the extended displacement discontinuity $||w|| + c_1||\varphi||$. For the same crack in a purely elastic medium, the boundary integral equation of the displacement discontinuity $\|W\|$ in the $z$-direction takes the same form [16]

$$\frac{E}{8\pi(1 - \nu^2)}\int_{S^+} \frac{||W||}{r^3}\,\mathrm{d}S = -t_z(x, y), \tag{11}$$