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A time-marching MFS scheme for heat conduction problems

Svilen S. Valtchev[a,*], Nilson C. Roberty[b]

[a]CEMAT and Departamento de Matemática, Instituto Superior Técnico, TULisbon, 1049-001 Lisboa, Portugal
[b]Programa de Engenharia Nuclear, Coordenação dos Cursos de Pós Graduação em Engenharia, Universidade Federal do Rio de Janeiro, Caixa Postal 68509, 21941-972, Rio de Janeiro, Brasil

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Abstract

In this work we consider the numerical solution of a heat conduction problem for a material with non-constant properties. By approximating the time derivative of the solution through a finite difference, the transient equation is transformed into a sequence of inhomogeneous Helmholtz-type equations. The corresponding elliptic boundary value problems are then solved numerically by a meshfree method using fundamental solutions of the Helmholtz equation as shape functions. Convergence and stability of the method are addressed. Some of the advantages of this scheme are the absence of domain or boundary discretizations and/or integrations. Also, no auxiliary analytical or numerical methods are required for the derivation of the particular solution of the inhomogeneous elliptic problems. Numerical simulations for 2D domains are presented. Smooth and non-smooth boundary data will be considered.

Keywords: Heat conduction problems; Meshfree method; Method of fundamental solutions; Finite differences

1. Introduction

Second order parabolic partial differential equations (PDEs) arise in a large number of mathematical and engineering problems, e.g. [1]. In this work we consider a general heat transfer problem of the form

$$\begin{cases} \rho\partial_t v - \nabla(k\nabla v) = f & \text{in } \Omega \times (0,T], \\ v = g_0 & \text{in } \bar{\Omega} \times \{0\}, \\ v = g_1 & \text{on } \Gamma_1 \times [0,T], \\ \partial_\nu v = g_2 & \text{on } \Gamma_2 \times [0,T], \\ \lambda v + \mu\partial_\nu v = g_3 & \text{on } \Gamma_3 \times [0,T], \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded, simply connected domain with a smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $T>0$ and ν is the unitary normal vector, exterior to the domain. We assume that the functions $f, \rho, k, g_0, g_1, g_2, g_3, \lambda, \mu$ possess the necessary regularity for the existence of a unique solution $v(x,t)$ of the above boundary value problem (BVP), e.g. [1]. Here, $k(x,t)>0$ is the thermal diffusivity, $\rho(x,t)>0$ is the thermal capacity and $v(x,t)$ is the temperature distribution on Ω at a time $t \in (0,T]$.

Using a Schrodinger transform of variables $v = k^{-1/2}u$, e.g. [2], Eq. (1) can be modified into the following gradient free form:

$$\begin{cases} \partial_t u - a\Delta u + bu = f & \text{in } \Omega \times (0,T], \\ u = g_0 & \text{in } \bar{\Omega} \times \{0\}, \\ u = g_1 & \text{on } \Gamma_1 \times [0,T], \\ \partial_\nu u = g_2 & \text{on } \Gamma_2 \times [0,T], \\ \lambda u + \mu\partial_\nu u = g_3 & \text{on } \Gamma_3 \times [0,T], \end{cases} \tag{2}$$

where $a{:=}\rho^{-1}k$, $b{:=}k^{1/2}(\partial_t k^{-1/2} + \rho^{-1}\Delta k^{1/2})$ and the functions f, g_0, g_1, g_2, g_3, λ, μ are redefined accordingly. Again, we consider a general case where the functions $a, b, f, g_1, g_2, g_3, \lambda$ and μ depend on the time and space variables x and t. To ensure that the PDE is uniformly parabolic, we also suppose that there exists a constant $\gamma>0$ such that $a(x,t) \geqslant \gamma > 0$, $\forall (x,t) \in \Omega \times [0,T]$. In this case (2) admits a unique solution.

Analytic solutions of (2) in some simple cases, like homogeneous problems for material with constant properties are available from the literature, e.g. [1,3]. In more

*Corresponding author.
E-mail addresses: ssv@math.ist.utl.pt (S.S. Valtchev), nilson@con.ufrj.br (N.C. Roberty).

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general settings, when irregular boundaries, non-trivial boundary conditions or non-smooth material properties are considered, numerical methods are the only mean of constructing an approximate solution of (2).

Traditionally, when the boundary value problems possess space and time dependent parameters, as well as an irregular geometry, the most appropriate method is the finite element method (FEM). For the solution of different classes of linear and non-linear PDEs, powerful multi-physics solvers, e.g. [4], are available from the software market. For self programmed software, it becomes difficult to generate the necessary domain and boundary discretizations and the difficulty increases in higher dimension problems, where mesh generation becomes the main computational challenge. Another commonly used but less sophisticated technique is based on the use of finite differences for the approximation of the time and space partial derivatives, e.g. [5]. The main drawback of the finite difference method is its relatively low accuracy, due to the numerical errors introduced in the process of the domain discretization. To overcome this kind of problems, meshfree techniques have been intensely researched by mathematicians and engineers in the last four decades. The most commonly used of these techniques are the method of fundamental solutions (MFS), e.g. [6,7] and methods based on the use of radial basis functions (RBF), e.g. [8].

Since its introduction, the MFS, [9] has been widely used by scientists for solving homogeneous problems, e.g. [6,10,11]. The classical MFS is based on the approximation of the solution of a BVP by a linear combination of fundamental solutions from the corresponding differential operator. The boundary conditions are then fitted by solving a collocation linear system. Note that the inversion of the collocation matrix may be avoided in some simple cases, e.g. a Dirichlet BVP for the Laplace equation in a disk. In these cases, by taking into account the circulant properties of the matrix, a FFT approach may be used to solve the system, e.g. [12,13].

In the last decade, an intensive research has been conducted regarding the application of the classical MFS to diffusion problems. Time-dependent fundamental solutions have been considered for some simple cases of homogeneous BVPs with constant coefficients [14] and also for unbounded and layered domains [15]. MFS has been successfully applied in these particular cases. However, in the general case of an inhomogeneous problem like (2), where the material properties are time and space dependent, a fundamental solution may not be available or may have only a local definition. Note that the Malgrange/Ehrenpreis existence theorem, [16,17], applies only for linear differential operators with constant coefficients.

Another relatively simple technique for solving a time dependent PDE is to transform it into a sequence of elliptic BVPs by using a finite difference (FD) to approximate the time derivative of the solution. The resulting inhomogeneous Helmholtz-type elliptic BVPs are then solved in two stages, first a particular solution is derived and then the corresponding homogeneous problem is solved. The main difficulty consists in finding a sufficiently accurate approximation of the particular solution, which may be a time consuming and mathematically elaborate process, e.g. [18]. Recent investigations, e.g. [18,19], have shown that RBFs, like polyharmonic splines or compactly supported RBFs, are a suitable choice as shape functions for the particular solution of Helmholtz-type equations. Finally, the homogeneous Helmholtz-type equation can be solved by classical FEM, BEM or FDM or through a meshfree method. RBF can be used again, or the MFS can be applied as shown in [7,20].

Recently, a modification of the classical MFS using fundamental solutions from an associated eigenvalue equation was introduced by Alves and Chen, cf. [21] for the solution of inhomogeneous Helmholtz BVPs. Once again two subproblems are solved, first the forcing term is approximated over a *span* of fundamental solutions with different source points and test frequencies. Using this, an approximation of the particular solution is obtained by rescaling the coefficients, at no extra computational cost. Finally, the associated homogeneous problem is solved using the classical MFS. The same idea was used (c.f. [22]) to introduce the plane waves method (PWM), which is a Trefftz type method, where the forcing term and the homogeneous solutions are approximated by plane waves with different directions of propagation and frequencies. The PWM was shown to be the asymptotic case of the MFS, avoiding, in this way, some of its floating point limitations. These two techniques were developed for the solution of Helmholtz equations with constant frequencies.

Our purpose in this work will be the coupling of an FD approximation with a recently developed modification of the MFS that can be applied for the approximate solution of a general second order elliptic BVP involving a PDE of the form $a(x)\,\Delta u(x) + b(x)u(x) = f(x)$. As discussed in [23], the total solution of this type of elliptic BVP can be approximated using fundamental solutions of the Helmholtz operator with different source points and test frequencies as shape functions. The mathematical justification of this method is based on density results, e.g. [10,24]. An important advantage is that the approximation is calculated through the solution of a single linear system, contrary to previous two stage meshfree methods where one system is solved for the particular solution and another for the homogeneous solution. The resulting FD–MFS hybrid method will be applied for the numerical solution of heat type transient problems for materials with non-constant properties.

In Section 3 we describe the proposed numerical scheme. In Section 4 some theoretical aspects will be addressed, namely, questions related with stability, consistency and convergence of the method. Finally, in Section 5, several 2D examples of problem (2) will be solved numerically using the new method.