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Differential quadrature Trefftz method for irregular plate problems

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ABSTRACT

Differential quadrature Trefftz method (DQTM) is developed to deal with plate problems defined in irregular domains. DQTM divides the solution into two parts, a particular solution for inhomogeneous biharmonic equation and the general solution for homogeneous biharmonic equation. For the former, differential quadrature method based on the interpolation of the highest derivative (DQIHD) is involved. For the latter, polynomial basis functions are adopted instead of fundamental solutions. We will show that DQTM not only keeps the advantages of traditional differential quadrature method (DQM), high efficiency and accuracy, but also has no difficulties to deal with geometrically irregular domains.

1. Introduction

Though bending problems of plates have been extensively studied, it is still very difficult to deal with arbitrary shapes and boundary conditions. In this paper, we try to offer a general method based on differential quadrature Trefftz method (DQTM). It combines differential quadrature method (DQM) [1–3] and Trefftz method [4–7], and the essence is to divide the solution into a particular solution for inhomogeneous biharmonic equation and the general solution for homogeneous biharmonic equation. For irregular plates, a large enough rectangular domain containing the original one is set as the computational domain. For a particular solution without boundary conditions, DQM based on the interpolation of the highest derivative (DQIHD) [8] is used, which is different from the traditional DQM. Because it does not involve the numerical differentiation process, the accuracy can be obviously heightened. For the general solution, we will express it by the linear combination of polynomial basis functions instead of fundamental solutions, so that there is no singularity [9,10]. Then the collocation method is used to determine the unknown coefficients.

On the one hand, this method can keep the advantages of DQM, such as high accuracy, efficiency and good convergence [3]. On the other hand, it is especially efficient for the plates whose shapes are irregular or whose boundary conditions are complex.

2. Methods

2.1. The linear bending problems of thin plates

According to Kirchhoff's three assumptions [11], the governing equation is a biharmonic equation. Here consider the standard form as follows:

$$\Delta^2 u = f, \tag{2.1}$$

where $f = q(x, y)/D$, $q(x, y)$ is a known transverse load function, D is the bending stiffness, u is the deflection w to solve. The boundary conditions in this paper involve clamped edges, simple supported edges and free edges, denoted by Γ_1, Γ_2 and Γ_3, respectively. So the boundary of Ω is $\Gamma = \partial\Omega = \Gamma_1 + \Gamma_2 + \Gamma_3$. Here the boundary conditions are expressed as follows:

$$\begin{cases} u = g_1, & \text{on} \quad \Gamma_1 \cup \Gamma_2, \\ Cu = g_2, & \text{on} \quad \Gamma_1, \\ Su = g_3, & \text{on} \quad \Gamma_2 \cup \Gamma_3, \\ Fu = g_4, & \text{on} \quad \Gamma_3, \end{cases} \tag{2.2}$$

where g_1, g_2, g_3 and g_4 are known functions and generally equal to 0. C, S and F are three operators defined as follows:

$$C = l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y},$$

$$S = (l^2 + \nu m^2)\frac{\partial^2}{\partial x^2} + (\nu l^2 + m^2)\frac{\partial^2}{\partial y^2} + (2-\nu)lm\frac{\partial^2}{\partial x \partial y},$$

$$F = -[(1-\nu)m^2 + 1]l\frac{\partial^3}{\partial x^3} + (1-\nu)(2l^2 - m^2 - 1)m\frac{\partial^3}{\partial x^2 \partial y} + (1-\nu)(2m^2 - l^2 - 1)l\frac{\partial^3}{\partial x \partial y^2} - [(1-\nu)l^2 + 1]m\frac{\partial^3}{\partial y^3}.$$

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It is easy to find that they are only related to the Poisson ratio of the plate v and $l = \cos(n, x)$, $m = \sin(n, x)$, where n is the outward normal direction.

2.2. DQIHD

For the boundary value problems (2.1) and (2.2), DQTM decomposes the solution as

$$u = u_p + u_h, \tag{2.3}$$

where u_p is a particular solution for (2.1) without considering the boundary conditions, so it is not unique. u_h is the general solution for homogeneous biharmonic equation:

$$\Delta^2 u = 0. \tag{2.4}$$

For a particular solution for (2.1), DQIHD is adopted here. It is introduced briefly in the following part and detailed in Ref. [8].

The essence of the traditional DQM is that the partial derivative of a function with respect to a variable can be approximated by a weighed sum of functional values at all discrete points in that direction. And the weighing coefficients depend only on the grid space. But DQIHD tries to use the values of the partial derivative at discrete points to express the functional values. Take the four-order ordinary differential equation for example. Firstly, $u^{(4)}$ can be approximated by the interpolation of the values at discrete grid points $\{x_i\}$ as

$$u^{(4)}(x) = \sum_{j=1}^{n} l_j(x) u^{(4)}(x_j), \tag{2.5}$$

where $l_j(x)$ is the basis function of Lagrange interpolation on the discrete points $\{x_i\}$. In this paper, we choose Gauss–Chebyshev points, expressed as

$$x_i = \cos\left(\frac{2i-1}{2n}\pi\right), \quad 1 \leqslant i \leqslant n. \tag{2.6}$$

Next, by integration we have

$$u^{(3)} = \int_0^x u^{(4)} \mathrm{d}s = \sum_{j=1}^{n} a_j u^{(4)}(x_j) + u^{(3)}(0), \tag{2.7}$$

where $a_j = \int_0^x l_j \, \mathrm{d}s$. Repeat this process until the order is 0. Denote

$$A = (a_{ij}),\ \ B = (b_{ij}),\ \ C = (c_{ij}),\ \ D = (d_{ij}),$$
$$a_{ij} = a_j(x_i),\ \ b_{ij} = b_j(x_i) = \int_0^{x_i} a_j(s)\mathrm{d}s,$$
$$c_{ij} = c_j(x_i) = \int_0^{x_i} b_j(s)\mathrm{d}s,\ \ d_{ij} = d_j(x_i) = \int_0^{x_i} c_j(s)\mathrm{d}s.$$

Then the process can be expressed as

$$U^{(4)} = \underline{E}\,\underline{U}, U^{(3)} = \underline{A}\,\underline{U}, U^{(2)} = \underline{B}\,\underline{U}, U^{(1)} = \underline{C}\,\underline{U}, U = \underline{D}\,\underline{U}, \tag{2.8}$$

where

$$\begin{aligned} \underline{U} &= [\,u(0) \quad u^{(1)}(0) \quad u^{(2)}(0) \quad u^{(3)}(0) \quad u^{(4)}(x_1) \quad \cdots \quad u^{(4)}(x_n)\,]^{\mathrm{T}} \\ \underline{E} &= [0,0,0,0,E],\ \ \underline{A} = [0,0,0,I,A],\ \ \underline{B} = [0,0,I,X,B], \\ \underline{C} &= [0,I,X,X^2/2,C],\ \ \underline{D} = [I,X,X^2/2,X^3/6,D], \end{aligned} \tag{2.9}$$

where E is an identity matrix, the terms like $u^{(p)}(0)$ $(0 \leqslant p \leqslant 3)$ are the values or derivatives of u on the boundary points, I, X, $X^2/2$, $X^3/6$ are column vectors making up of the values of functions 1, x, $x^2/2$, $x^3/6$ at $\{x_i\}$. Notice that matrices with underlines are no longer square.

Extend to two-dimensional cases. Since the biharmonic operator contains three terms: $\partial^4/\partial x^4$, $\partial^4/\partial x^2\partial y^2$ and $\partial^4/\partial y^4$ and the highest partial derivatives along x and y are both of four order, all of them can be expressed by $(u_p)_{x^4y^4}$ via numerical integration instead. Here $\underline{U}$ is denoted as the value matrix of $(u_p)_{x^4y^4}$ and called the highest derivative of U_p. $\underline{U}$ can be expressed as follows:

$$\underline{U} = \begin{pmatrix} u(0,0) & u_{0,1}(0,0) & u_{0,2}(0,0) & u_{0,3}(0,0) & u_{0,4}(0,y_1) & \cdots & u_{0,4}(0,y_n) \\ u_{1,0}(0,0) & u_{1,1}(0,0) & u_{1,2}(0,0) & u_{1,3}(0,0) & u_{1,4}(0,y_1) & \cdots & u_{1,4}(0,y_n) \\ u_{2,0}(0,0) & u_{2,1}(0,0) & u_{2,2}(0,0) & u_{2,3}(0,0) & u_{2,4}(0,y_1) & \cdots & u_{2,4}(0,y_n) \\ u_{3,0}(0,0) & u_{3,1}(0,0) & u_{3,2}(0,0) & u_{3,3}(0,0) & u_{3,4}(0,y_1) & \cdots & u_{3,4}(0,y_n) \\ u_{4,0}(x_1,0) & u_{4,1}(x_1,0) & u_{4,2}(x_1,0) & u_{4,3}(x_1,0) & u_{4,4}(x_1,y_1) & \cdots & u_{4,4}(x_1,y_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{4,0}(x_n,0) & u_{4,1}(x_n,0) & u_{4,2}(x_n,0) & u_{4,3}(x_n,0) & u_{4,4}(x_n,y_1) & \cdots & u_{4,4}(x_n,y_n) \end{pmatrix}, \tag{2.10}$$

where $u_{s,t}(x,\ y)$ $(0 \leqslant s,\ t \leqslant 4)$ is short for $u_{x^sy^t}(x,y)$. Generally, the boundary values$u_{x^sy^t}(x_i,0)$ and$u_{x^sy^t}(0,y_j)$ are given via the boundary conditions. But in this paper, they are also unknowns.

DQIHD uses the equations as follows for a particular solution:

$$\underline{E}_x\,\underline{U}\,\underline{D}_y^{\mathrm{T}} + 2\underline{B}_x\,\underline{U}\,\underline{B}_y^{\mathrm{T}} + \underline{D}_x\,\underline{U}\,\underline{E}_y^{\mathrm{T}} = \underline{F}. \tag{2.11}$$

It can be transformed into

$$(\underline{D}_y \otimes \underline{E}_x + 2\underline{B}_y \otimes \underline{B}_x + \underline{E}_y \otimes \underline{D}_x)\mathrm{Vec}(\underline{U}) = \mathrm{Vec}(\underline{F}), \tag{2.12}$$

where $\otimes$ is Kronecker-product, $\mathrm{Vec}(\cdot)$ means to reset the matrix as a column vector. (2.12) may have many solutions. The least-square method is used here to get one and according to the relationship between $\underline{U}$ and $U_{x^sy^t}$, all derivatives are solved.

Compared with the traditional DQM, DQIHD involves no numerical differentiation, which is very sensitive to even a small level of errors. It is not only for particular solutions. For bending moments and shearing forces which also involve derivatives, the process is almost the same. If the traditional DQM is chosen for particular solutions, numerical differentiation will be unavoidable, because we have to let C, S, F act on the particular solution u_p. So DQIHD can avoid numerical differentiation in two places.

2.3. General solution for homogeneous biharmonic equation

The general solution u_h can be expressed by the linear combination of a series of basis functions $\{u_k^*\}_{k=1,2\ldots}$, i.e.

$$u_h = \sum_k \beta_k u_k^*. \tag{2.13}$$

And general solutions of bending moments and shearing forces can be expressed by linear combinations of relevant derivatives of basis functions, with the same coefficients.

There can be different basis functions. Usually fundamental solutions are adopted. But here we choose the polynomial basis functions, because they have no singularity. They can be obtained similarly according to the following theorem.

Theorem 4.1. [10] *A biharmonic function u(x, y) in a plane simply connected domain Ω certainly can be expressed as*

$$u(x,y) = Re[\bar{z}\varphi(z) + \psi(z)], \tag{2.14}$$

where $\varphi(z)$ and $\psi(z)$ are two analytic function, $z = x+yi$; conversely, for two arbitrary analytic functions $\varphi(z)$ and $\psi(z)$ in Ω, the function u(x, y) given by (2.14) *is certainly a biharmonic function.*

Let Ω be a plane bounded domain. In order to obtain a series of simple biharmonic functions, we can first let $\varphi(z)$ and $\psi(z)$ as polynomials of z, then substitute them into (2.14) and take the real parts. Distinctly, all of the harmonic functions are also biharmonic