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Integral equation formulation for thin shells—revisited

Subrata Mukherjee*

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, NY 14853, USA

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Abstract

While the matter of solving problems in fracture mechanics with boundary integral equations (BIEs) has received considerable attention, the "conjugate" problem of thin shells has received less. The latter problem is revisited in this work. The attempt here is to clarify certain issues that were not fully addressed in earlier publications. In particular, the issue of consistency of the relevant regularized boundary integral equations, when collocated at corresponding points on two sides of a thin shell, is addressed carefully. Some comments on the corresponding BIE formulation for fracture mechanics problems are made at the end of the paper in order to compare and contrast these two problems.

Keywords: Boundary integral equations; Cracks; Thin shells; Fracture mechanics

1. Introduction

The problem of a crack in an (infinite or finite) linear elastic medium has received considerable attention in the boundary integral equation (BIE) literature (please see [1] and the references therein). The failure of the standard BIE in the limit of an infinitely thin crack is well known; as is the idea of employing the hypersingular BIE (HBIE) to solve this problem. The conjugate problem of a thin shell has been solved with an elasticity BIE approach by Liu [2]. Certain issues related to this formulation have been discussed in [3] (see, also, [1]).

The formulation presented in [3] is revisited in this paper. Regularized displacement BIEs, collocated at points $\mathbf{x}^+$ (on the upper) and $\mathbf{x}^-$ (on the lower) surfaces of a thin shell, are carefully derived. It is shown that consistency of these equations requires, as expected, that the displacement must be continuous across a thin shell. Similarly, one concludes from the consistency requirement of the corresponding regularized stress BIEs that the stress must also be continuous across a shell. Of course, such continuity in these fields is to be expected. It is interesting, however, to demonstrate that consistency of the relevant regularized BIEs demands continuity of the relevant fields.

Finally, it is shown that while these BIEs in the limit of a thin crack are well posed and solvable (this is well known), those for thin shells are not! Fortunately, however, problems involving thin shells of finite thickness have been successfully solved numerically by Liu [2] using the displacement BIE in its standard form—i.e. by modeling the displacements separately on the two shell surfaces rather than just their difference, together with careful evaluation of the nearly singular integrals that arise in his formulation.

2. Mathematical preliminaries

Formulae for integral of the traction kernel $\mathbf{T}$ over a boundary element, and for the solid angle subtended by a boundary element at a point, are very important for the rest of this paper. Hence, these issues are addressed first.

The standard BIEs for three-dimensional (3-D) linear elasticity involve the traction kernel. This kernel has the form [1, Eq. (1.19)]:

$$T_{ik} = -\frac{1}{8\pi(1-\nu)r^2}\left[\{(1-2\nu)\delta_{ik} + 3r_{,i}r_{,k}\}\frac{\partial r}{\partial n} + (1-2\nu)(r_{,i}n_k - r_{,k}n_i)\right]. \quad (1)$$

*Tel.: +1 607 255 7143; fax: +1 607 255 2011.
E-mail address: sm85@cornell.edu.

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In the above $\mathbf{r}(\mathbf{x},\mathbf{y}) = \mathbf{y} - \mathbf{x}$ (with $\mathbf{x}$ a source and $\mathbf{y}$ a field point), $r = |\mathbf{r}|$, $r_{,k} = \partial r/\partial y_k = (y_k - x_k)/r$, $\mathbf{n}$ is the unit outward normal to the boundary at $\mathbf{y}$, $\boldsymbol{\delta}$ is the Kronecker delta and ν is the Poisson's ratio of the elastic material.

Inside approach: The following finite part integrals (FP integrals $⨎$ in the sense of Mukherjee [4]) of the kernel $\mathbf{T}$ (see Fig. 1 below and (3.26) in [1]) are noted below:

$$⨎^{(I)}_{\hat{S}^+} T_{ik}(\mathbf{x},\mathbf{y})\,\mathrm{d}S(\mathbf{y}) = -g^{(a)}_{ik}(\mathbf{x}) - \frac{\Omega^{(I)}(\hat{S}^+,\mathbf{x})}{4\pi}\delta_{ik}, \tag{2}$$

$$⨎^{(I)}_{\hat{S}^-} T_{ik}(\mathbf{x},\mathbf{y})\,\mathrm{d}S(\mathbf{y}) = -g^{(c)}_{ik}(\mathbf{x}) - \frac{\Omega^{(I)}(\hat{S}^-,\mathbf{x})}{4\pi}\delta_{ik}, \tag{3}$$

where

$$g^{(a)}_{ik}(\mathbf{x}) = \frac{1-2\nu}{8\pi(1-\nu)}\varepsilon_{ik\ell}\oint^{(a)}_{\hat{L}^+}\frac{1}{r(\mathbf{x},\mathbf{y})}\,\mathrm{d}z_\ell + \frac{\varepsilon_{k\ell m}}{8\pi(1-\nu)}\oint^{(a)}_{\hat{L}^+}\frac{r_{,i}(\mathbf{x},\mathbf{y})r_{,\ell}(\mathbf{x},\mathbf{y})}{r(\mathbf{x},\mathbf{y})}\,\mathrm{d}z_m. \tag{4}$$

In (2)–(4), $\mathbf{x}$ is a boundary point that can be $\mathbf{x}^+$ or $\mathbf{x}^-$ (see Fig. 1). The surface element $\hat{S}^+ \subset S^+$ is a neighborhood of $\mathbf{x}^+ \in S^+$. It is noted that, strictly speaking, the integral on the left-hand side of (2) is strongly singular only for $\mathbf{x} = \mathbf{x}^+$ and nearly strongly singular for $\mathbf{x} = \mathbf{x}^-$. Therefore, its FP designation is strictly valid only in the limit of a very thin shell. (A similar comment also applies to (3).)

Also, $\boldsymbol{\varepsilon}$ is the alternating tensor, $z_k = y_k - x_k$, and the line integrals over $\hat{L}^+$ are evaluated in the anti-clockwise sense when viewed from above. The expression for $g^{(c)}_{ik}$ is the same as that on the right-hand side of (4), except that the integral is now evaluated over $\hat{L}^-$ in the clockwise sense when viewed from above. In the case of a very thin shell (shell thickness $h \to 0$), $\mathbf{g}^{(a)}(\mathbf{x}) \approx -\mathbf{g}^{(c)}(\mathbf{x})$.

The superscripts (I) in (2) and (3) indicate approach to $\mathbf{x}^+$ from a point $\boldsymbol{\xi}$ *inside* the shell (see Fig. 1). The solid angle $\Omega^{(I)}$, subtended by $\hat{S}^+$ and $\hat{S}^-$ at $\mathbf{x}^+$, are given by (see Fig. 2 and [5])

$$\Omega^{(I)}(\hat{S}^+,\mathbf{x}^+) = ⨎_{\hat{S}^+}\frac{\mathbf{r}(\mathbf{x}^+,\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{r^3(\mathbf{x}^+,\mathbf{y})}\,\mathrm{d}S(\mathbf{y}) = \int_0^{2\pi}[1-\cos(\psi(\theta))]\,\mathrm{d}\theta, \tag{5}$$

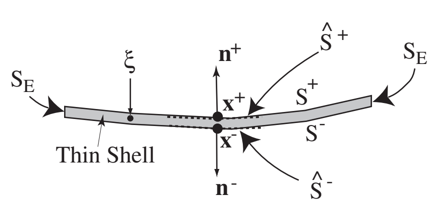

Fig. 1. Geometry of a thin shell. The unit normal point away from the shell.

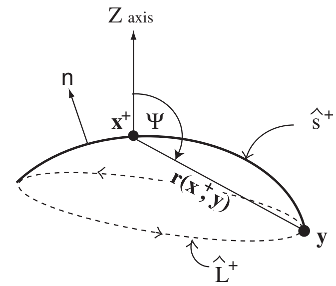

Fig. 2. Surface element $\hat{S}^+$ with bounding contour $\hat{L}^+$.

$$\Omega^{(I)}(\hat{S}^-,\mathbf{x}^-) = ⨎_{\hat{S}^-}\frac{\mathbf{r}(\mathbf{x}^-,\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{r^3(\mathbf{x}^-,\mathbf{y})}\,\mathrm{d}S(\mathbf{y}) = \int_0^{2\pi}[1+\cos(\psi(\theta))]\,\mathrm{d}\theta. \tag{6}$$

The solid angle Ω was first expressed in this way by Liu [2]. A Cartesian coordinate system $OXYZ$ is chosen with the origin at the source point (inside the shell) such that the positive Z-axis intersects the appropriate surface ($\hat{S}^+$ for (5) and $\hat{S}^-$ for (6)). In Fig. 2, ψ is the angle between the positive Z-axis and $\mathbf{r}(\mathbf{x}^+,\mathbf{y})$ with $\mathbf{y} \in \hat{L}^+$, and θ is the angle between the X-axis and the projection of $\mathbf{r}(\mathbf{x}^+,\mathbf{y})$ in the X–Y plane (see Figs. 5 and 6 in [2]).

It is noted here that for a very thin shell $\mathbf{x}^+ \approx \mathbf{x}^-$, so that, one has $\Omega^{(I)}(\hat{S}^+,\mathbf{x}^+) \approx \Omega^{(I)}(\hat{S}^+,\mathbf{x}^-)$ and $\Omega^{(I)}(\hat{S}^-,\mathbf{x}^+) \approx \Omega^{(I)}(\hat{S}^-,\mathbf{x}^-)$.

Outside approach: The situation is very interesting if a boundary point $\mathbf{x}$ is approached by a point $\boldsymbol{\xi}$ *outside* the shell. The equations corresponding to (2) and (3) now are

$$⨎^{(O)}_{\hat{S}^+} T_{ik}(\mathbf{x},\mathbf{y})\,\mathrm{d}S(\mathbf{y}) = -g^{(a)}_{ik}(\mathbf{x}) - \frac{\Omega^{(O)}(\hat{S}^+,\mathbf{x})}{4\pi}\delta_{ik}, \tag{7}$$

$$⨎^{(O)}_{\hat{S}^-} T_{ik}(\mathbf{x},\mathbf{y})\,\mathrm{d}S(\mathbf{y}) = -g^{(c)}_{ik}(\mathbf{x}) - \frac{\Omega^{(O)}(\hat{S}^-,\mathbf{x})}{4\pi}\delta_{ik}. \tag{8}$$

It is noted that the value of the tensor $\mathbf{g}$ is independent of how the point $\mathbf{x}$ is approached. The value of the solid angle, however, does depend on both the direction of the normal to the surface as well as the direction of approach [5]. For a very thin shell, one has (see Fig. 2)

$$\Omega^{(O)}(\hat{S}^+,\mathbf{x}^+) = -\int_0^{2\pi}[1+\cos(\psi(\theta))]\,\mathrm{d}\theta = -\Omega^{(O)}(\hat{S}^-,\mathbf{x}^+), \tag{9}$$

$$\Omega^{(O)}(\hat{S}^+,\mathbf{x}^-) = \int_0^{2\pi}[1-\cos(\psi(\theta))]\,\mathrm{d}\theta = -\Omega^{(O)}(\hat{S}^-,\mathbf{x}^-). \tag{10}$$

An interesting observation: In this discussion, $\mathbf{x}$ is a boundary point which can be either $\mathbf{x}^+$ or $\mathbf{x}^-$ (see Fig. 1). Further, very thin shells are considered here.

It is noted that (5), (6) and the last two lines of the paragraph "Inside approach" in Section 2 imply that

$$\Omega^{(I)}(\hat{S}^+,\mathbf{x}) \not\approx -\Omega^{(I)}(\hat{S}^-,\mathbf{x}). \tag{11}$$