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Topological sensitivity analysis in the context of ultrasonic non-destructive testing

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ABSTRACT

This paper deals with the use of the topological derivative in detection problems involving waves. In the first part, a framework to carry out the topological sensitivity analysis in this context is proposed. Arbitrarily shaped holes and cracks with Neumann boundary condition in 2 and 3 space dimensions are considered. In the second part, a numerical example concerning the treatment of ultrasonic probing data in metallic plates is presented. With moderate noise in the measurements, the defects (air bubbles) are detected and satisfactorily localized by means of a single sensitivity computation.

1. Introduction

Inspection problems can generally be seen as shape inversion problems. If techniques borrowed from shape optimization are now commonly accepted as good theoretical candidates to address shape inversion problems, their applications to inspection problems such as non-destructive testing or medical imaging are today relatively restricted. The main reason is that, in such problems, the possibility to handle topology changes is crucial. Therefore the use of the topological derivative concept, which directly deals with the variable "topology", seems to be particularly well-suited. We recall the basic principles of this approach, introduced by Schumacher [1], Sokolowski and Zochowski [2] in structural optimization. Consider a cost function $\mathscr{J}(\Omega) = J_\Omega(u_\Omega)$ where u_Ω is the solution of a system of partial differential equations defined in the domain $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, a point $x_0 \in \Omega$ and a fixed open and bounded subset ω of $\mathbb{R}^N$ containing the origin. The "topological asymptotic expansion" is an expression of the form

$$\mathscr{J}(\Omega \backslash \overline{(x_0 + \rho\omega)}) - \mathscr{J}(\Omega) = f(\rho)g(x_0) + o(f(\rho)), \tag{1}$$

where $f(\rho)$ is a positive function tending to zero with ρ. Therefore, to minimize $\mathscr{J}(\Omega)$, we have interest to remove matter where the "topological gradient" (also called "topological derivative", or "topological sensitivity") g is negative.

A general framework enabling to calculate the topological asymptotic expansion for a large class of shape functionals has been worked out by Masmoudi [3]. It is based on an adaptation of the adjoint method and a domain truncation technique providing an equivalent formulation of the PDE in a fixed function space. Using this framework, Garreau, Guillaume, Masmoudi and Sididris [4–6] have obtained the topological asymptotic expansions for several problems associated with linear and homogeneous differential operators. For such operators, but with a different approach, more general shape functionals are considered in [7]. The link between the shape and the topological derivatives has been established by Feijóo et al. [8,9]. This gives rise to a generic method for deriving the latter. However, it seems rather restricted to circular or spherical holes. For the first time a topological sensitivity analysis for a non-homogeneous operator was performed in [10]. The case of a circular hole with a Dirichlet condition imposed on its boundary was considered. For completeness, we point out that extensive research efforts using related techniques have been done in the context of reconstruction problems from boundary measurements (see e.g. [11–15] among others). In contrast to our approach, they do not deal with an explicit cost function. Instead, the sensitivity of the PDE solution u_Ω at the location of the measurements (or its integral against special test functions) is computed. Then the data are interpreted by signal processing methods. In addition, those works are focused on the detection of inhomogeneities, which means that the material density inside the inclusions is non-zero.

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The problem of interest in this paper is related to non-destructive testing by means of ultrasounds in the context of elastodynamics. The governing equations at a fixed frequency involve a non-homogeneous differential operator of the form

$$u \mapsto \operatorname{div}(A\nabla u) + k^2 u, \tag{2}$$

where A is a symmetric positive definite tensor. For such a problem, the topological asymptotic expansion is determined in dimensions 2 and 3 with respect to the creation of an arbitrarily shaped hole and an arbitrarily shaped crack on which a Neumann boundary condition is prescribed. For the sake of simplicity, the analysis is presented for the Helmholtz operator ($A = I$), but it applies similarly to any operator of the form (2). We introduce an adjoint method that takes into account the variation of the function space, so that a domain truncation is not needed. This formalism brings several technical simplifications, notably for the study of criteria depending explicitly on Ω, for which the truncation necessitates to transport the cost function in the fixed domain (see [5]). Similar results have been obtained in [16]. However, the analysis is not done there in a rigorous mathematical framework. Furthermore, the crack problem is not addressed and less general cost functionals are considered.

The rest of the present paper is organized as follows. The adjoint method is presented in Section 2. The framework of the study is described in Section 3. The topological asymptotic analysis for a hole and a crack is carried out in Sections 4 and 5, respectively, the intermediate proofs being reported in Section 8. The case of some particular cost functions is examined in Section 6. Section 7 is devoted to numerical experiments that highlight the relevance of the topological sensitivity approach for non-destructive testing applications.

2. An appropriate adjoint method

In this section, the adjoint method is generalized to a class of problems for which the state variable belongs to a function space that depends on the control variable. Let $(\mathscr{V}_\rho)_{\rho \geqslant 0}$ be a family of Hilbert spaces on the complex field such that

$$\mathscr{V}_0 \subset \mathscr{V}_\rho \quad \forall \rho \geqslant 0.$$

For all $\rho \geqslant 0$, let a_ρ be a sesquilinear and continuous form on $\mathscr{V}_\rho$ and let l_ρ be a semilinear and continuous form on $\mathscr{V}_\rho$. We assume that, for all $\rho \geqslant 0$, the variational problem

$$\begin{cases} u_\rho \in \mathscr{V}_\rho, \\ a_\rho(u_\rho, v) = l_\rho(v) \quad \forall v \in \mathscr{V}_\rho \end{cases} \tag{3}$$

admits a unique solution. We make the following assumption.

Hypothesis 1. For all $\rho \geqslant 0$, there exist on $\mathscr{V}_\rho$ a sesquilinear and continuous form $\tilde{a}_\rho$ and a semilinear and continuous form $\tilde{l}_\rho$ such that

$$\tilde{a}_\rho(u_0, v) = \tilde{l}_\rho(v) \quad \forall v \in \mathscr{V}_\rho. \tag{4}$$

Consider now a criterion $j(\rho) = J_\rho(u_\rho) \in \mathbb{R}$ where, for all $\rho \geqslant 0$, the function J_ρ is differentiable in the following sense: there exists a linear and continuous form on $\mathscr{V}_\rho$ denoted by L_ρ such that

$$J_\rho(u_0 + h) - J_\rho(u_0) = \Re L_\rho(h) + o(\|h\|_{\mathscr{V}_\rho}). \tag{5}$$

The notation $\Re z$ stands for the real part of the complex number z. Furthermore, we assume that, for all $\rho \geqslant 0$, the adjoint problem

$$\begin{cases} v_\rho \in \mathscr{V}_\rho, \\ a_\rho(u, v_\rho) = -L_\rho(u) \quad \forall u \in \mathscr{V}_\rho \end{cases} \tag{6}$$

admits an unique solution. We make the two following additional hypotheses.

Hypothesis 2. There exist two complex numbers δ_a and δ_l and a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, when ρ tends to zero

$$f(\rho) \longrightarrow 0,$$

$$(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) = f(\rho)\delta_a + o(f(\rho)),$$

$$(l_\rho - \tilde{l}_\rho)(v_\rho) = f(\rho)\delta_l + o(f(\rho)).$$

Hypothesis 3. There exists a real number δ_J such that

$$J_\rho(u_\rho) - J_0(u_0) = \Re L_\rho(u_\rho - u_0) + f(\rho)\delta_J + o(f(\rho)).$$

Then, the asymptotic expansion of $j(\rho)$ is given by the theorem below.

Theorem 1. *If Hypotheses* 1–3 *hold*, *then*

$$j(\rho) - j(0) = f(\rho)\Re(\delta_a - \delta_l + \delta_J) + o(f(\rho)).$$

Proof. Using Eq. (3) and Hypothesis 1, we obtain

$$\begin{aligned} j(\rho) - j(0) = {} & J_\rho(u_\rho) - J_0(u_0) + \Re(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) \\ & + \Re a_\rho(u_\rho - u_0, v_\rho) - \Re(l_\rho - \tilde{l}_\rho)(v_\rho). \end{aligned}$$

Hypotheses 2 and 3 and Eq. (6) yield

$$\begin{aligned} j(\rho) - j(0) = {} & \Re L_\rho(u_\rho - u_0) + f(\rho)\delta_J + f(\rho)\Re\delta_a \\ & - \Re L_\rho(u_\rho - u_0) - f(\rho)\Re\delta_l + o(f(\rho)), \end{aligned}$$

from which we deduce the announced result. □

3. The topological sensitivity problem

3.1. Problem formulation

Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, $N = 2$ or 3, with smooth boundary Γ. We assume for simplicity that Γ is piecewise of class $\mathscr{C}^\infty$, but this hypothesis could be considerably weakened. We consider a function $u_0 \in H^1(\Omega)$ satisfying the state equations

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0 & \text{in } \Omega, \\ \partial_n u_0 = S u_0 + \sigma & \text{on } \Gamma. \end{cases} \tag{7}$$

Here, $\mathbf{n}$ denotes the outward unit normal of Γ, $k \in \mathbb{C}$, $S \in \mathscr{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$, namely the space of continuous linear maps from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$, and $\sigma \in H^{-1/2}(\Gamma)$. For a given $x_0 \in \Omega$ and a small parameter $\rho > 0$, we denote by Ω_ρ the perturbed domain. We shall address two situations.

- In the case of a perforation, we consider a fixed open and bounded subset ω of $\mathbb{R}^N$ containing the origin and whose boundary Σ is the union of two graphs of functions of class $\mathscr{C}^1$ from $\mathbb{R}^{N-1}$ into $\mathbb{R}$ (this technical hypothesis could also be weakened). We define $\omega_\rho = x_0 + \rho\omega$, $\Sigma_\rho = \partial\omega_\rho$ and $\Omega_\rho = \Omega \backslash \overline{\omega_\rho}$ (see Fig. 1(a)).

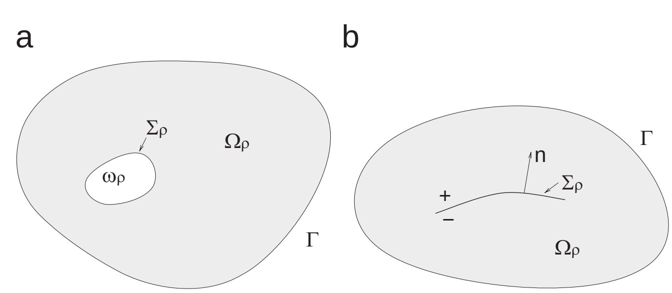

Fig. 1. The perturbed domain: (a) perforated domain, (b) cracked domain.