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A dual reciprocity boundary element formulation using the fractional step method for the incompressible Navier–Stokes equations

Chang-Yong Choi [a,*], Elias Balaras [b]

[a] Department of Mechanical and Automotive Engineering, Jeonju University, Jeonju 560-759, Korea
[b] Fischell Department of Bioengineering, University of Maryland, College Park, MD 20742, USA

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ABSTRACT

This paper presents a dual reciprocity boundary element solution method for the unsteady Navier–Stokes equations in two-dimensional incompressible flow, where a fractional step algorithm is utilized for the time advancement. A fully explicit, second-order, Adams–Bashforth scheme is used for the nonlinear convective terms. We performed numerical tests for two examples: the Taylor–Green vortex and the lid-driven square cavity flow for Reynolds numbers up to 400. The results in the former case are compared to the analytical solution, and in the latter to numerical results available in the literature. Overall the agreement is excellent demonstrating the applicability and accuracy of the fractional step, dual reciprocity boundary element solution formulations to the Navier–Stokes equations for incompressible flows.

1. Introduction

The overall accuracy of many grid-based approaches in computational fluid dynamics (CFD), such as the finite element method (FEM), finite difference method (FDM), or finite volume method (FVM), depends to a large extend, on the quality of the computational mesh. In complex geometries the generation of high-quality grids is difficult and usually has an adverse impact on the accuracy and stability of the computations. An attractive alternative to classical grid-based approaches is the family of boundary element methods (BEM) [1]. BEM greatly simplify the grid generation procedure, since the problem dimensions are reduced, and only the boundary is discretized. In the past, BEM have been extensively used in linear and nonlinear heat transfer problems, where today have reached a stage of maturity and are routinely used in the field [2–4].

In the case of fluid flow problems, however, progress has not been as rapid and most efforts have been concentrated in steady flows. Grigorev [5], for example, proposed the BEM formulation using the SIMPLE algorithm [6] to solve the Navier–Stokes equations for steady, incompressible, laminar flows. Since the resulting boundary integral equations are nonlinear, an iterative scheme with under-relaxation was utilized. Rahaim and Kassab [7] developed the dual reciprocity boundary element method (DRBEM) based on an on iterative pressure correction scheme. They demonstrated the accuracy of the approach in steady, viscous, incompressible flow in a channel for several Reynolds numbers. Sarler and Kuhn [8] developed a DRBEM formulation based on the SIMPLE algorithm, which utilized augmented thin-plate, spline functions as the global interpolation functions. Camacho and Barbosa [9] applied a similar scheme to convective heat transfer problems. The limitations of the SIMPLE algorithm, however, in unsteady flow problems are well established, and fractional step methods are much more popular in the CFD community [10–14]. Eldho and Young [15] proposed an alternative DRBEM approach, which is based on the velocity–vorticity formulation for steady, incompressible flows. The Poisson and vorticity transport equations in the resulting scheme were solved iteratively using DRBEM and combined to determine the velocity and vorticity vectors. The accuracy of the method was demonstrated for lid-driven cavity at low Reynolds numbers.

In the present paper we have developed a BEM for the unsteady Navier–Stokes equations, where a fractional step algorithm is utilized for the time advancement. The overall approach is based on the classical projection method proposed by Chorin [16]. A fully explicit, second-order, Adams–Bashforth scheme is used for the nonlinear convective terms. The boundary-only discretization, which is the key element of the DRBEM is preserved by taking the domain integral to the boundary and removing the need for a complicated domain discretization with the reciprocity principle [17]. To demonstrate the accuracy and efficiency of the method we will present two examples: the Taylor–Green vortex and the lid-driven square cavity flow. The results in the former case are compared to the analytical solution,

* Corresponding author. Tel.: +82 63 220 2615; fax: +82 63 220 2750.
E-mail addresses: cychoi@jj.ac.kr (C.-Y. Choi), balaras@umd.edu (E. Balaras).

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and in the latter to numerical results available in the literature. In the next sections the overall formulation and solution method are discussed in detail. The results and conclusions will follow.

2. Fractional step method

The dynamics of viscous incompressible flow with constant density and viscosity are determined by the momentum conservation equation,

$$\rho\frac{\partial \mathbf{V}}{\partial t}+\rho\mathbf{V}\cdot\nabla\mathbf{V}=-\nabla P+\mu\nabla^2\mathbf{V}, \tag{1}$$

together with mass conservation equation,

$$\nabla\cdot\mathbf{V}=0, \tag{2}$$

where $\mathbf{V}$ is velocity vector, t is time, ρ is the fluid density, μ is the viscosity and P is the pressure. In the framework of a fractional step method, a predictor step is first taken, and an approximate velocity field (denoted by tilde from now on), which is not divergence free is obtained for the time increment, Δt, between time-step m and m+1:

$$\rho\frac{\tilde{\mathbf{V}}^{(m+1)}-\mathbf{V}^{(m)}}{\Delta t}=-\rho\mathbf{V}\cdot\nabla\mathbf{V}+\mu\nabla^2\mathbf{V}. \tag{3}$$

Then the predicted velocity is projected into a divergence free field using,

$$\rho\frac{\mathbf{V}^{(m+1)}-\tilde{\mathbf{V}}^{(m+1)}}{\Delta t}=-\nabla p, \tag{4}$$

which can be written as

$$\mathbf{V}^{(m+1)}=\tilde{\mathbf{V}}^{(m+1)}-\frac{\Delta t}{\rho}\nabla p, \tag{5}$$

where the pressure, p, is a pseudo-pressure and not the actual pressure. This pseudo-pressure field that is used to perform the projection can be calculated from Eq. (4) as follows:

$$\nabla\cdot\left(\rho\frac{\mathbf{V}^{(m+1)}-\tilde{\mathbf{V}}^{(m+1)}}{\Delta t}\right)=\nabla\cdot(-\nabla p), \tag{6}$$

which can be written as

$$\nabla^2 p=\frac{\rho}{\Delta t}\nabla\cdot\tilde{\mathbf{V}}^{(m+1)}. \tag{7}$$

In unsteady flows, an advantage of the fractional step method compared conventional SIMPLE based algorithms is the fact that it uncouples the velocity from the pressure and therefore sub-iterations within each time-step are not necessary for convergence.

3. DRBEM formulations for the predicted velocity field

To implement the standard DRBEM procedure the predictor step given by Eq. (3) can be rewritten as

$$\nabla^2\mathbf{V}=\frac{1}{\mu}\left(\rho\frac{\tilde{\mathbf{V}}^{(m+1)}-\mathbf{V}^{(m)}}{\Delta t}+\rho\mathbf{V}\cdot\nabla\mathbf{V}\right)=\mathbf{b}, \tag{8}$$

where the source term of the Poisson-like equation is concisely expressed by vector $\mathbf{b}$. Let the general variable φ to represent the velocity $\mathbf{V}$ and use the variable $\psi(=\partial\varphi/\partial\mathrm{n})$ as its normal derivative. Then applying the usual boundary element technique to Eq. (8) based on the use of the fundamental solution of the Laplace equation φ^* and its normal derivative ψ^* with the reciprocity principle (Green's theorem) [17], a boundary integral equation in a domain Ω with boundary Γ can be deduced to

$$c_i\varphi_i+\int_\Gamma\varphi\psi^* d\Gamma=\int_\Gamma\psi\varphi^* d\Gamma+\int_\Omega b\varphi^* d\Omega. \tag{9}$$

In the DRBEM, the domain integral is taken to the boundary by expanding the source term $\mathbf{b}$ as its values at each node j using a set of interpolating functions f_j as

$$b\cong\sum_{j=1}^{N+L}\alpha_j f_j, \tag{10}$$

where N+L is the number of boundary nodes plus internal points, α_j is a set of initially unknown coefficients. The interpolating functions f_j and the particular solutions $\hat{\varphi}_j$ are linked so that the domain integral can be transferred to boundary as

$$\nabla^2\hat{\varphi}_j=f_j. \tag{11}$$

Then by substituting Eq. (11) into Eq. (10) and applying integration by parts twice for the domain integral term, the dual reciprocity boundary integral equation can be derived as

$$c_i\varphi_i+\int_\Gamma\varphi\psi^* d\Gamma-\int_\Gamma\psi\varphi^* d\Gamma=\sum_{j=1}^{N+L}\alpha_j\left(c_i\hat{\varphi}_{ij}+\int_\Gamma\hat{\varphi}_j\psi^* d\Gamma-\int_\Gamma\hat{\psi}_j\varphi^* d\Gamma\right). \tag{12}$$

For the numerical solution of the integral Eq. (12), the matrix form of DRBEM is obtained by using the standard BEM scheme of boundary discretization with linear elements [5]:

$$[H]\{\varphi\}-[G]\{\psi\}=\sum_{j=1}^{N+L}\alpha_j([H]\{\hat{\varphi}_j\}-[G]\{\hat{\psi}_j\})=([H][\hat{\Phi}]-[G][\hat{\Psi}])\{\alpha\}, \tag{13}$$

where $[H]$, $[G]$ represent the influence coefficient matrices resulting from the discretization process, and $[\hat{\Phi}]$, $[\hat{\Psi}]$ denote the matrices containing the vectors $\{\hat{\varphi}_j\}$, $\{\hat{\Psi}_j\}$ for the collocation points j, respectively. They are the functions of geometry and can be readily evaluated with the use of quadrature. Here the expansion coefficients α_j can be calculated from the chosen interpolating function f_j as

$$\{\alpha\}=[F]^{-1}\{b\}, \tag{14}$$

where the square matrix $[F]$ is formed by vectors of the interpolating functions f_j. Therefore Eq. (13) becomes

$$[H]\{\varphi\}-[G]\{\psi\}=[S]\{b\}, \tag{15}$$

and matrix $[S]$ is given by

$$[S]=([H][\hat{\Phi}]-[G][\hat{\Psi}])[F]^{-1}. \tag{16}$$

Since two-dimensional problems are considered in the present study, the general variable φ can be expressed as u for x component velocity and v for y component velocity. Then expressions of vector $\{b\}$ are:

$$b_x=\frac{1}{\mu}\left[\frac{\rho}{\Delta t}(u^{m+1}-u^m)+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\right] \tag{17}$$

and

$$b_y=\frac{1}{\mu}\left[\frac{\rho}{\Delta t}(v^{m+1}-v^m)+\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\right]. \tag{18}$$

In the time marching procedure, the nonlinear convective terms in the above equations can be treated implicitly or explicitly. An explicit, second-order, Adams–Bashforth scheme is employed in