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Hyper singular boundary element formulation for the Grad-Shafranov equation as an axisymmetric problem

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ABSTRACT

The Grad-Shafranov equation describes the magnetic flux distribution of plasma in an axisymmetric system such as a tokamak-type nuclear fusion device. This paper presents a scheme to solve the hyper singular boundary integral equation (HBIE) corresponding to this Grad-Shafranov equation. All hyper and strong singularities caused by differentials of the complete elliptic integrals have been regularized up to the level of the Cauchy principal value integral. Test calculations commonly using discontinuous boundary elements have been made to compare the HBIE solutions with the solutions of the standard boundary integral equation (SBIE).

1. Introduction

The magnetohydrodynamic (MHD) equilibrium of plasma in an axisymmetric (r, z) system such as a 'tokamak' nuclear fusion device is described by the Grad-Shafranov equation

$$-\left\{r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}\right\}\psi=\mu_0 r j_\varphi \tag{1}$$

in terms of magnetic flux ψ and the toroidal component of of the plasma current j_φ [1]. The quantity μ_0 means the permeability in a vacuum. The boundary element method (BEM) [2] was applied to solving this equation [3–5]. In this application, the inhomogeneous current term $\mu_0 r j_\varphi$ is expanded into a 2-D polynomial

$$\mu_0 r j_\varphi \approx \sum_{l,\,m}\alpha_{l,\,m} r^l z^m \quad (l\geqslant 0,\ \ m\geqslant 0). \tag{2}$$

The domain integral caused by $\mu_0 r j_\varphi$ is transformed into an equivalent boundary one, using a particular solution $\varphi^{(l,m)}$ corresponding to each term in the above polynomial [3,5] and applying Green's second identity. The boundary integral equation for the plasma boundary Γ has the form

$$c_i\psi_i-\int_\Gamma\left(\frac{\psi^*}{r}\frac{\partial\psi}{\partial n}-\frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma = \sum_{l,\,m}\alpha_{l,\,m}\left\{c_i\varphi_i^{(l,\,m)}-\int_\Gamma\left(\frac{\psi^*}{r}\frac{\partial\varphi^{(l,\,m)}}{\partial n}-\frac{\varphi^{(l,\,m)}}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma\right\} \tag{3}$$

with the fundamental solution ψ^*. Itagaki et al. [4] also applied the above boundary element formulation to an inverse analysis where the plasma current density profile was reconstructed from signals of magnetic sensors located outside the plasma.

Apart from the above 'standard' boundary integral equation (SBIE), the hyper singular boundary integral equation (HBIE) [6–10] arises when one takes a gradient of the SBIE. The authors have a future plan to introduce an HBIE approach into the above inverse analysis as an alternative to the SBIE or as a part of the combination of these two equations.

The HBIE corresponding to the Grad-Shafranov equation has never been solved before. As this equation is for axisymmetric geometries, the fundamental solution and its derivatives are written mathematically in quite complicated forms, all of which contain the complete elliptic integrals (see appendix). Thus one needs to pay careful attention to their singularities when manipulating this type of HBIE.

In the present paper the HBIE for the Grad-Shafranov equation is regularized in a similar manner as that Mansur et al. [10] used for the HBIE to solve the Laplace equation. A distinctive feature of the present work is that one must deal also with the polynomial expanded source. Even this inhomogeneous source generates a boundary integral, which also contains a hyper singular kernel.

Section 2 describes the process to transform the original HBIE into a form that is convenient to remove the singularities. The resultant boundary integral equation is given in Section 2.5. In Section 3, all boundary integral terms in the resultant equation are further rearranged in such a way that each term converges to a

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finite value. Discontinuous boundary elements are commonly used for all numerical examples given in Section 4, where the HBIE solutions are compared with the SBIE solutions.

2. Hyper singular boundary integral equation

One here starts with the standard boundary integral equation for an internal point i

$$\psi_i - \int_\Gamma \left(\frac{\psi^*}{r}\frac{\partial\psi}{\partial n} - \frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\right) d\Gamma = \sum_{l,\,m}\alpha_{l,\,m}\left\{\varphi_i^{(l,\,m)} - \int_\Gamma \left(\frac{\psi^*}{r}\frac{\partial\varphi^{(l,\,m)}}{\partial n} - \frac{\varphi^{(l,\,m)}}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma\right\}, \tag{4}$$

by substituting $c_i = 1.0$ into Eq. (3). The HBIE is given by differentiating Eq. (4) at the point i along an arbitrary direction $\mathbf{m} = (m_r, m_z)$. Using the notation $\partial/\partial m = \mathbf{m}\cdot\nabla$, the HBIE is written in the form

$$\frac{\partial\psi_i}{\partial m} - \int_\Gamma \left(\frac{1}{r}\frac{\partial\psi^*}{\partial m}\frac{\partial\psi}{\partial n} - \frac{\psi}{r}\frac{\partial^2\psi^*}{\partial m\,\partial n}\right) d\Gamma = \sum_{l,\,m}\alpha_{l,\,m}\left\{\frac{\partial\varphi_i^{(l,\,m)}}{\partial m} - \int_\Gamma \left(\frac{1}{r}\frac{\partial\psi^*}{\partial m}\frac{\partial\varphi^{(l,\,m)}}{\partial n} - \frac{\varphi^{(l,\,m)}}{r}\frac{\partial^2\psi^*}{\partial m\,\partial n}\right)d\Gamma\right\}. \tag{5}$$

Consider a small semicircle of radius ε on the boundary as depicted in Fig. 1. The source point i is assumed to be at the center of this semicircle and afterwards the radius ε is reduced to zero. In the following discussion, $\mathbf{x} = (r, z)$ denotes an arbitrary point along the boundary, while $\boldsymbol{\xi} = (a, b)$ means the source point i, i.e., the singular point.

Considering that the boundary is divided into Γ_ε and $\Gamma - \Gamma_\varepsilon$, Eq. (5) is rewritten in the form

$$\begin{aligned}&\frac{\partial\psi(\boldsymbol{\xi})}{\partial m} - \sum_{l,\,m}\alpha_{l,\,m}\frac{\partial\varphi^{(l,\,m)}(\boldsymbol{\xi})}{\partial m} \\ &\quad - \int_{\Gamma-\Gamma_\varepsilon}\left(\frac{1}{r}\frac{\partial\psi^*}{\partial m}\frac{\partial\psi(\mathbf{x})}{\partial n} - \frac{\psi(\mathbf{x})}{r}\frac{\partial^2\psi^*}{\partial m\,\partial n}\right)d\Gamma \\ &\quad + \sum_{l,\,m}\alpha_{l,\,m}\int_{\Gamma-\Gamma_\varepsilon}\left(\frac{1}{r}\frac{\partial\psi^*}{\partial m}\frac{\partial\varphi^{(l,\,m)}(\mathbf{x})}{\partial n} - \frac{\varphi^{(l,\,m)}(\mathbf{x})}{r}\frac{\partial^2\psi^*}{\partial m\,\partial n}\right)d\Gamma \\ &= \int_{\Gamma_\varepsilon}\left(\frac{1}{r}\frac{\partial\psi^*}{\partial m}\frac{\partial\psi(\mathbf{x})}{\partial n} - \frac{\psi(\mathbf{x})}{r}\frac{\partial^2\psi^*}{\partial m\,\partial n}\right)d\Gamma \\ &\quad - \sum_{l,\,m}\alpha_{l,\,m}\int_{\Gamma_\varepsilon}\left(\frac{1}{r}\frac{\partial\psi^*}{\partial m}\frac{\partial\varphi^{(l,\,m)}(\mathbf{x})}{\partial n} - \frac{\varphi^{(l,\,m)}(\mathbf{x})}{r}\frac{\partial^2\psi^*}{\partial m\,\partial n}\right)d\Gamma.\end{aligned} \tag{6}$$

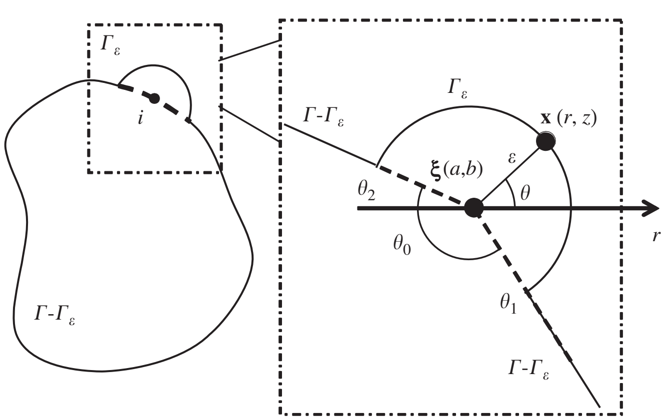

Fig. 1. Boundary surface augmented by a small semicircle of radius ε.

2.1. Limiting forms of the fundamental solution and its derivatives

The fundamental solution ψ^* satisfies a subsidiary equation

$$-\Delta^*\psi^* = r\delta_i, \tag{7}$$

where δ_i means $\delta(r-a)\delta(z-b)$ with the spike at the point i having the coordinates (a, b). The mathematical form of ψ^* is given by [3–5]

$$\psi^* = \frac{\sqrt{ar}}{\pi k}\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right] \tag{8}$$

with

$$k^2 = \frac{4ar}{(r+a)^2 + (z-b)^2}, \tag{9}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively. When the field point (r, z) approaches the source point (a, b), $K(k)$ and $E(k)$ can be approximated as [11]

$$K(k) \approx \frac{1}{2}\log\left(\frac{16}{1-k^2}\right) = \log\frac{1}{\varepsilon} + \log\left(4\sqrt{\varepsilon^2 + 4ar}\right) \tag{10a}$$

and

$$E(k) \approx 1, \tag{10b}$$

where $\varepsilon = \sqrt{(r-a)^2 + (z-b)^2}$. Starting with Eqs. (10a) and (10b), the authors derived the limiting forms of the fundamental solution and its derivatives when $\varepsilon \to 0$ (the mathematical forms of the derivatives of ψ^* are listed in appendix). In this process, the relationships, $r - a = \varepsilon\cos\theta$ and $z - b = \varepsilon\sin\theta$ were used. The results are shown below.

First, the limit of the fundamental solution is given by

$$\psi^* \to \frac{a}{2\pi}\log\frac{1}{\varepsilon} + \frac{a}{2\pi}\log 8a - \frac{a}{\pi}. \tag{11}$$

Also, the derivatives of ψ^* with respect to a and b approach

$$\frac{\partial\psi^*}{\partial a} \to \frac{\log 8a - 1}{4\pi} - \frac{1}{4\pi}\log\,\varepsilon + \frac{a\,\cos\,\theta}{2\pi}\frac{1}{\varepsilon} \tag{12a}$$

and

$$\frac{\partial\psi^*}{\partial b} \to \frac{a\,\sin\,\theta}{2\pi}\frac{1}{\varepsilon} \tag{12b}$$

when $\varepsilon \to 0$, i.e., $r \to a$ and $z \to b$. As a linear combination of Eqs. (12a) and (12b), the derivatives of ψ^* along the direction $\mathbf{m}$ takes the limit

$$\begin{aligned}\frac{\partial\psi^*}{\partial m} &= m_r\frac{\partial\psi^*}{\partial a} + m_z\frac{\partial\psi^*}{\partial b} \to \frac{m_r(\log 8a - 1)}{4\pi} - \frac{m_r}{4\pi}\log\,\varepsilon \\ &\quad + \frac{a(m_r\,\cos\,\theta + m_z\,\sin\,\theta)}{2\pi}\frac{1}{\varepsilon} \\ &\equiv D_0(\theta) + D_1(\theta)\log\,\varepsilon + D_2(\theta)/\varepsilon.\end{aligned} \tag{13}$$

Next, one investigates the limit of $\partial^2\psi/\partial m\,\partial n$. Based on the following four limits:

$$\frac{\partial}{\partial a}\left(\frac{\partial\psi^*}{\partial r}\right) \to \frac{(2 - \log 8a) + \log 8a\,\cos 2\theta}{16\pi a} + \frac{1 - \cos 2\theta}{16\pi a}\log\,\varepsilon + \frac{\cos\,\theta}{2\pi}\frac{1}{\varepsilon} - \frac{a\,\cos 2\theta}{2\pi}\frac{1}{\varepsilon^2}, \tag{14a}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial\psi^*}{\partial z}\right) \to \frac{(\log 8a + 1)\sin 2\theta}{16\pi a} - \frac{\sin 2\theta}{16\pi a}\log\,\varepsilon - \frac{\sin\,\theta}{4\pi}\frac{1}{\varepsilon} - \frac{a\,\sin 2\theta}{2\pi}\frac{1}{\varepsilon^2}, \tag{14b}$$