Engineering Analysis with Boundary Elements 33 (2009) 858–865

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A Galerkin boundary node method for biharmonic problems

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ARTICLE INFO

Article history:
Received 30 April 2008
Accepted 4 November 2008
Available online 6 January 2009

Keywords:
Meshless
Galerkin boundary node method
Biharmonic equation
Moving least-squares
Boundary integral equation

ABSTRACT

A Galerkin boundary node method (GBNM) is developed in this paper for solving biharmonic problems. The GBNM combines an equivalent variational form of boundary integral formulations for governing equations with the moving least-squares approximations for construction of the trial and test functions. In this approach, only a nodal data structure on the boundary of a domain is required. In addition, boundary conditions can be implemented accurately and the system matrices are symmetric. The convergence of this method and numerical examples are given to show the efficiency.

1. Introduction

Boundary integral equations (BIEs) and boundary element methods (BEMs) [1,2] are attractive computational techniques for linear and exterior problems as they can reduce the dimensionality of the original problem by one. Especially for exterior problems, the use of classical methods, such as finite difference or finite element methods (FEMs) [3], requires discretization of the entire exterior, whereas with the BEM only the surface needs to be discretized. However, the BEM still requires boundary discretization, which may cause some inconvenience in the implementation, such as attacking complicated boundary problems and moving boundary problems. For the sake of avoiding meshing, a new type of method called meshfree or meshless method has been developed in recent years [4,5]. That method does not require a mesh to discretize the problem domain, and the approximate solution is constructed entirely based on a set of scattered nodes.

The meshless methods can be divided into two categories: the domain type and the boundary type. Several domain type meshless methods, such as the element free Galerkin method (EFGM) [4,6], the reproducing kernel particle method (RKPM) [7], the moving least-square (MLS) reproducing kernel method [8], the finite point method [9] and the h–p meshless method [10] are very promising methods, and their mathematical background were well investigated.

The boundary type meshless methods are developed by the combination of the meshless idea with BIEs, such as the boundary node method (BNM) [11,12], the hybrid boundary node method (HBNM) [13–15], the regular hybrid boundary node method (RHBNM) [16,17], the boundary point interpolation method [18], the boundary cloud method (BCM) [19], the boundary knot method [20], and the boundary particle method [21]. Compared with domain type meshless methods, this type of approaches have a well-known dimensionality of the BEM, thus they have been proposed and achieved remarkable progress in solving a broad class of boundary value problems. Despite their popularity, there exist some problems related to their efficient implementation. Among these there are difficulties in satisfying boundary conditions when the shape functions lack the delta function property, the system matrices of many boundary type methods are non-symmetric, and the theoretical basis of these methods is just being studied and far from completion.

In this paper, a Galerkin boundary node method (GBNM) is developed for the interior and exterior biharmonic problems. The GBNM represents a coupling between the MLS approach [4,22] and a variational formulation of BIEs. The MLS approximation is used to construct the trial and test functions. In the GBNM, boundary conditions can be satisfied accurately via multiplying the MLS shape function and integrating on the boundary. Besides, the system matrices are symmetric. This property of symmetry can be an added advantage in coupling the GBNM with the FEM or other established meshless methods such as the EFGM.

The rest of this paper is outlined as follows. Section 2 presents the MLS approximation and its properties. In Section 3, a detailed numerical implementation of the GBNM is described for solving biharmonic problems. Section 4 provides the convergence analysis of this method in Sobolev spaces. Numerical examples are given in Section 5. Section 6 contains some conclusions.

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doi:10.1016/j.enganabound.2008.11.002

2. The MLS approximation

In the MLS method, the numerical approximation starts from a cluster of scattered nodes instead of elements. This section gives a brief summary of the MLS approximation, of which excellent illustrations can be seen in Refs. [4,22].

Let Γ be a smooth, simple closed curve in the plane and let Ω and Ω' denote its interior and exterior, respectively. Let $Q_N := \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ denote an arbitrarily chosen set of N nodes. The set Q_N is used for defining a finite open covering $\{\Re_i\}_{i=1}^N$ of Γ composed of N balls $\Re_i$ centered at the points $\mathbf{x}_i$, where $\Re_i$ is the support domain of $\mathbf{x}_i$. Besides, let w_i, $i = 1, 2, \ldots, N$, denote nonnegative weight functions that belong to the space $C_0^\alpha(\Re_i)$, $\alpha \geqslant 0$, and satisfy $\sum_{i=1}^N w_i(\mathbf{x}) = 1$, $\forall \mathbf{x} \in \Gamma$.

Assume that $\mathbf{x}(s) \in \Gamma$, the MLS approximation for a given function v is

$$v(\mathbf{x}) \approx \mathscr{M}v(\mathbf{x}) := \sum_{i=1}^{N} \Phi_i(\mathbf{x}) v_i \tag{1}$$

where $\mathscr{M}$ is an approximation operator

$$\Phi_i(\mathbf{x}(s)) := \sum_{j=0}^{\beta} P_j(s)[\mathbf{A}^{-1}(s)\mathbf{B}(s)]_{ji} \tag{2}$$

and the matrixes $\mathbf{A}(s)$ and $\mathbf{B}(s)$ being defined by

$$\mathbf{A}(s) = \sum_{i=1}^{N} w_i(s)\mathbf{P}(s_i)\mathbf{P}^{\mathrm{T}}(s_i) \tag{3}$$

$$\mathbf{B}(s) = [w_1(s)\mathbf{P}(s_1), w_2(s)\mathbf{P}(s_2), \ldots, w_N(s)\mathbf{P}(s_N)] \tag{4}$$

in which s is a curvilinear co-ordinate on Γ, $\mathbf{P}(s)$ is a vector of the polynomial basis, $\beta + 1$ is the number of terms of the monomials.

Assumption. For our subsequent error analysis, we impose the following conditions:

A1. There exists a nonnegative integer $\gamma \leqslant \alpha$ such that the MLS shape functions $\Phi_i(\mathbf{x}) \in C^\gamma(\Gamma)$ and the boundary Γ is a C^γ curve.
A2. There is a constant h such that the radii of any boundary point's support domain is less than h.
A3. There exist nonnegative integers $K_1(\mathbf{x}) \geqslant \beta$ and $K_2(\mathbf{x})$ such that for any $\mathbf{x} \in \Gamma$, there are at least $K_1(\mathbf{x})$ boundary nodes, and at most $K_2(\mathbf{x})$ boundary nodes lie on the support domain of $\mathbf{x}$.
A4. There are constants $C_{\Phi 1}$ and $C_{\Phi 2}$ independent of h such that $C_{\Phi 1} h^{-j} \leqslant \|D^j \Phi_i(\mathbf{x})\|_{L^\infty(\Gamma)} \leqslant C_{\Phi 2} h^{-j}$, $0 \leqslant j \leqslant \gamma$, $1 \leqslant i \leqslant N$.

Remark 2.1. Assumption (A3) is quite natural since, otherwise, as the number of boundary nodes lie on a local area increases, the shape functions tend to be more and more linearly dependent in the local area. Additionally, as indicated by Duarte and Oden [10], a necessary condition for the moment matrix $\mathbf{A}(s)$ to be invertible is that there are at least β nodes covered in the support domain of every sample point $\mathbf{x}(s) \in \Gamma$.

Notation 2.1. In what follows we will use the notation $\Re^i$ for the set of boundary points whose support domain including the boundary node $\mathbf{x}_i$, $1 \leqslant i \leqslant N$.

It should be pointed out that for different boundary point, $\mathbf{x}$, the support domain varies from point to point, hence $\Re^i \equiv \Re_i$ if and only if the radii of the support domain is a constant for all boundary points.

According to weight functions $w_i(s) \in C_0^\alpha(\Re_i)$ and the condition of (A1), we can easily deduce that the MLS shape functions have compact supports, i.e.:

Proposition 2.1. $\Phi_i(\mathbf{x}) \in C_0^\gamma(\Re^i)$, $1 \leqslant i \leqslant N$.

Proposition 2.2 (*Liu et al.* [8]). $\sum_{i=1}^N D^j \Phi_i(s)(s_i - s)^k = k!\delta_{jk}$, $0 \leqslant j \leqslant \alpha$, $0 \leqslant k \leqslant \beta$.

The following theorem gives an approximation estimate for the MLS approximations, which is central to the convergence proof of the presented GBNM.

Theorem 2.1. *Assume that* $v(\mathbf{x}) \in H^{m+1}(\Gamma)$. *Let* $\mathscr{M}v(\mathbf{x}) = \sum_{i=1}^N \Phi_i(\mathbf{x})v_i$, *then*

$$\|v(\mathbf{x}) - \mathscr{M}v(\mathbf{x})\|_{H^k(\Gamma)} \leqslant Ch^{m+1-k}\|v(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \quad 0 \leqslant k \leqslant m \leqslant \gamma \tag{5}$$

where C *is a constant independent of* h, *and* $H^k(\Gamma)$ *means the Sobolev space of functions defined on the curve* Γ [23].

This result was proved by Han and Meng [7] in the context of the RKPM. The proof of the theorem above exactly along the same lines and we shall omit the proof.

3. GBNM for biharmonic problems

3.1. Galerkin procedures

We consider the following interior and exterior biharmonic problems:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \text{ or } \Omega' = \mathbb{R}^2/\Omega \\ u = u_0 & \text{on } \Gamma = \partial\Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = g & \text{on } \Gamma \end{cases} \tag{6}$$

where $u_0 \in H^{3/2}(\Gamma)$ and $g \in H^{1/2}(\Gamma)$ are prescribed functions, and $\mathbf{n} = (n_1, n_2)$ is the outward normal to the boundary.

The biharmonic problem (6) is important in the modeling of many engineering applications such as bending of thin plate and flow of viscous fluid. In the case of the exterior problem, we append to problem (6) the following condition at infinity [24]:

$$u(\mathbf{x}) = O(|\mathbf{x}|) \quad \text{as } |\mathbf{x}| \to \infty \tag{7}$$

to be sure of the uniqueness of the solution.

Let q be the jump of traces of the function $\partial\Delta u/\partial\mathbf{n}$ across the boundary Γ, and φ be the jump through Γ of Δu. Then according to the classical results of partial differential equation, problem (6) admits only one solution in $H^2(\Omega) \times W_0^2(\Omega')$ [25,26], i.e.,

$$u(\mathbf{y}) = -\int_\Gamma q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})\,dS_\mathbf{x} + \int_\Gamma \varphi(\mathbf{x})\frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{x}}dS_\mathbf{x} + p(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2 \tag{8}$$

where $u^*(\mathbf{x},\mathbf{y}) = (1/8\pi)|\mathbf{x}-\mathbf{y}|^2 \ln|\mathbf{x}-\mathbf{y}|$ is the fundamental solution for the biharmonic operator, $p(\mathbf{y}) \in \mathbb{P}_1$ with $\mathbb{P}_1$ denoting the set of polynomials of degree no more than one, and $W_0^2(\Omega')$ is a weighted Sobolev space [27]

$$W_0^2(\Omega') := \{u \in \mathscr{D}'(\Omega') | (1+r^2)^{(|\lambda|-2)/2}(\ln(2+r^2))^{-1}D^\lambda u \in L^2(\Omega'), \\ |\lambda| = 0, 1; D^\lambda u \in L^2(\Omega'), |\lambda| = 2\} \tag{9}$$

with $\lambda = (\lambda_1, \lambda_2)$, $|\lambda| = \lambda_1 + \lambda_2$, and $r = |\mathbf{x}|$ represents the distance from the origin to the point $\mathbf{x} \in \mathbb{R}^2$.

According to the boundary conditions of problem (6), we get the following BIEs:

$$\begin{cases} u_0(\mathbf{y}) = -\int_\Gamma q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})\,dS_\mathbf{x} + \int_\Gamma \varphi(\mathbf{x})\dfrac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{x}}dS_\mathbf{x} + p(\mathbf{y}) \\ g(\mathbf{y}) = -\int_\Gamma q(\mathbf{x})\dfrac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}}dS_\mathbf{x} + \int_\Gamma \varphi(\mathbf{x})\dfrac{\partial^2 u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{x}\partial \mathbf{n}_\mathbf{y}}dS_\mathbf{x} + \dfrac{\partial p(\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}}, \end{cases} \mathbf{y} \in \Gamma \tag{10}$$