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The method of fundamental solutions for problems of free vibrations of plates

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Abstract

In this paper a new boundary method for problems of free vibrations of plates is presented. The method is based on mathematically modelling of the physical response of a system to external excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. So, contrary to the traditional scheme, the method described does not involve evaluation of determinants of linear systems. The method shows a high precision in simply and doubly connected domains. The results of the numerical experiments justifying the method are presented.

Keywords: Free vibrations; Plates; Method of fundamental solutions; Multipole expansion

1. Introduction

The free vibrations of an isotropic thin elastic plate are described by the following equation:

$$\rho h \frac{\partial^2 u}{\partial t^2} + D\nabla^4 u, \quad u = u(\mathbf{x}, t), \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2. \tag{1}$$

Here u is the normal displacement of the middle surface of the plate, ρ, h and D are the volume density, the thickness and the rigidity of the plate.

Considering harmonic vibrations

$$u(\mathbf{x}, t) = w(\mathbf{x}) \exp(\mathrm{i}\omega t)$$

the governing equation can be written in the following dimensionless form

$$\nabla^4 w - k^4 w = 0, \quad k^4 = \frac{\rho h a^4 \omega^2}{D}, \tag{2}$$

where a is a typical linear size of the plate. The problem of free vibration is to find the real k for which there exist non-null functions w verifying (2) and some homogeneous boundary conditions:

$$\mathbf{B}[w] = 0, \quad \mathbf{x} \in \partial\Omega. \tag{3}$$

The operator of the boundary conditions $\mathbf{B}[\ldots]$ will be specified below.

The problems (2), (3) is a classical problem of mathematical physics. Apart from a few analytically solvable cases [1–3], there is no general solution of this problem. Therefore, a large number of numerical methods have been developed for many practical problems. The usual approach for eigenvalue problems with a positive defined operator is to use the Rayleigh minimal principle. See [4–6] for more details and references. Then, using an approximation for w with a finite number of free parameters, one gets the same problem in a finite-dimensional subspace which can be solved by a standard procedure of linear algebra, e.g., see [7,8]. The global basis functions [9–11] as well as finite elements [12,13] are used for this approximation.

Recently, some new powerful numerical techniques have been developed in this field. These are the differential quadrature methods proposed by Bellman and coworkers in 1972 [14], its recent version—the generalized differential quadrature (GDQ) approach [15,16] and the discrete singular convolution (DSC) algorithm which can be regarded as a local spectral method [17,18].

The boundary methods [19], in particular, the method of fundamental solutions (MFS) [20,21] are convenient in application to the problems (2), (3).

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In the framework of the boundary methods a general approach to solving these problems is as follows. First, using an integral representation of w in the BEM, or an approximation over fundamental solutions in MFS, one gets a homogeneous linear system $\mathscr{A}(k)\mathbf{q} = \mathbf{0}$ with matrix elements depending on the wave number k. To obtain the non-trivial solution the determinant of this matrix must be zero:

$$\det[\mathscr{A}(k)] = 0. \tag{4}$$

To get the eigenvalues this equation must be investigated analytically or numerically. This technique is described in [22–26] with more details. In the two latest papers there is a complete bibliography on the subject considered.

Another technique is proposed in [27–29]. This is a mathematical model of physical measurements when the resonance frequencies of a system are determined by the amplitude of response to some external excitation.

Let us consider the eigenvalue problem:

$$L[w] + \lambda w = 0, \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2, \quad B[w] = 0, \quad \mathbf{x} \in \partial\Omega. \tag{5}$$

The method presented is as follows. Let us extend the operator of the problem from the initial domain Ω into a more wider Ω_0. In particular case $\Omega_0 = \mathscr{R}^2$. Let $w_p(\mathbf{x})$ be a particular solution of the PDE

$$L[w] + \lambda w = f(\mathbf{x}), \quad \mathbf{x} \in \Omega_0,$$

where $f(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega \subset \Omega_0$. If w_h is the solution of the boundary value problem

$$L[w_h] + \lambda w_h = 0, \quad \mathbf{x} \in \Omega,$$
$$B[w_h(\mathbf{x})] = -B[w_p(\mathbf{x})], \quad \mathbf{x} \in \partial\Omega,$$

then, the sum $w(\mathbf{x}, \lambda) = w_h + w_p$ satisfies (5). Let $F(\lambda)$ be some norm of the solution w. This function of λ has extremums at the eigenvalues and, under some conditions described below, can be used for their determining.

The outline of this paper is as follows. The main algorithm is described in Section 2. In Section 3, we give numerical examples to illustrate the method presented for simply and multiple connected domains. In particular, the case of doubly connected region with the inner region of vanishing maximal dimension which is important for technical applications is considered here.

2. The main algorithm

2.1. 1D case

For the sake of simplicity, let us consider 1D problem of free vibrations of a homogeneous beam with simply supported endpoints (SS conditions).

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho S}\frac{\partial^4 u}{\partial x^4} = 0, \quad 0 \leqslant x \leqslant l, \tag{6}$$

$$u(0, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = 0, \quad u(l, t) = \frac{\partial^2 u}{\partial x^2}(l, t) = 0,$$

where E is Young's modulus, ρ is density, S and I are the area and moment of inertia of the cross section. Let us consider the harmonic vibration

$$u(x, t) = w(x)\mathrm{e}^{\mathrm{i}\omega t}.$$

The eigenvalue problem, can be written in the dimensionless form as follows:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - k^4 w = 0, \tag{7}$$

$$w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0, \tag{8}$$

where

$$k^4 = \frac{\rho S l^4 \omega^2}{EI}. \tag{9}$$

It can be proved that k is a dimensionless value. The problem (7), (8) has a well-known solution:

$$k_n = n\pi, \quad \varphi_n(x) = \sin(n\pi x).$$

On the other hand, the differential operator of the problem can written as a product

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4} - k^4 = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right) \equiv \mathscr{L}_2(k)\mathscr{L}_1(k).$$

Let us assume that $k \neq 0$, then, the two singular solutions corresponding these two operators are

$$\Phi_1(x, \xi) = \exp(\mathrm{i}k|x - \xi|), \quad \Phi_2(x, \xi) = \exp(k|x - \xi|). \tag{10}$$

The MFS solution of (7), (8) can be written in the following way:

$$\begin{aligned} w &= q_1\Phi_1(x, \xi_1) + q_2\Phi_2(x, \xi_1) + q_3\Phi_1(x, \xi_2) + q\Phi_2(x, \xi_2) \\ &= q_1\mathrm{e}^{\mathrm{i}k(x-\xi_1)} + q_2\mathrm{e}^{k(x-\xi_1)} + q_3\mathrm{e}^{-\mathrm{i}k(x-\xi_2)} + q_4\mathrm{e}^{-k(x-\xi_2)}, \end{aligned}$$

where $\xi_1 < 0$ and $\xi_2 > 1$ are the positions of the MFS source points.

Using the boundary conditions (8) and setting equal to zero the determinant of resulting linear system we get

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ \mathrm{e}^{\mathrm{i}k} & \mathrm{e}^{k} & \mathrm{e}^{-\mathrm{i}k} & \mathrm{e}^{-k} \\ -\mathrm{e}^{\mathrm{i}k} & \mathrm{e}^{k} & -\mathrm{e}^{-\mathrm{i}k} & \mathrm{e}^{-k} \end{vmatrix} = 0,$$

or after simple transforms:

$$(\mathrm{e}^{\mathrm{i}k} - \mathrm{e}^{-\mathrm{i}k})(\mathrm{e}^{k} - \mathrm{e}^{-k}) = 0.$$

We get the wave numbers k_n as solutions: $\sin(k) = 0$, or $k = n\pi$. Thus, MFS gives the exact solution. Note that in multidimensional cases such computations are not so simple and are time consuming.

According to the technique presented we solve the inhomogeneous problem:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - k^4 w = f(x), \quad w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0. \tag{11}$$

Here $x \in [A, B]$, where $[A, B] \supset [0, 1]$ is a large enough